

CSE203B Convex Optimization: Lecture 3: Convex Function

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Outlines

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1. Definitions: Convex Function vs Convex Set

Theorem: Given $S = \{x | f(x) \leq b\}$

If function $f(x)$ is convex, then S is a convex set.

Proof: We prove by the definition of convex set.

For every $u, v \in S$, i. e. $f(u) \leq b, f(v) \leq b$,

We want to show that $\alpha u + \beta v \in S, \forall \alpha + \beta = 1, \alpha, \beta \geq 0$.

We have

$$\begin{aligned} f(\alpha u + \beta v) &\leq \alpha f(u) + \beta f(v) \quad (f \text{ is convex}) \\ &\leq \alpha b + \beta b \quad (\alpha, \beta \geq 0) \\ &= (\alpha + \beta) \cdot b = b \quad (\alpha + \beta = 1) \end{aligned}$$

Thus $\alpha u + \beta v \in S$

Remark: Convex function \Rightarrow Convex Set

$$f(x) \leq b \quad \Rightarrow \text{Convex Set}$$

$$f(x) \geq b \quad \Rightarrow ?$$

1. Convex Function Definitions: Examples

$f: R^n \rightarrow R$ is convex if $dom f$ is a convex set and

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$
$$\forall x, y \in dom f, 0 \leq \theta \leq 1$$

Example on R :

Convex Functions

Affine: $ax + b$ on R for any $a, b \in R$

Exponential: e^{ax} for any $a \in R$

Power: x^α on R_{++} for $\alpha \geq 1$ or $\alpha \leq 0$
 $|x|^p$ on R for $p \geq 1$

Concave Functions

Affine: $ax + b$ on R for any $a, b \in R$

Power: x^α on R_{++} for $0 \leq \alpha \leq 1$

Logarithm: $\log x$ on R_{++}

1. Convex Function Definitions: Examples

Example on R^n :

Affine: $f(x) = a^T x + b$

Norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$;

$$\|x\|_\infty = \max_k |x_k|$$

Example on $R^{m \times n}$:

Affine: $f(X) = \text{tr}(A^T X) = \sum_{i=1}^m \sum_{j=1}^n A_{ij} x_{ij}$

Spectral (max singular value):

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

1. Convex Function Definitions: Examples

Concave Functions:

Log Determinant: $f(X) = \log \det X$, $\text{dom } f = S_{++}^n$

Proof: Let $g(t) = f(X + tV)$ ($V \in S^n$)

$$\begin{aligned} g(t) &= \log \det (X + tV) = \log \det X + \log \det (I + tX^{-\frac{1}{2}}VX^{-\frac{1}{2}}) \\ &= \log \det X + \sum_{i=1}^n \log(1 + t\lambda_i) \end{aligned}$$

λ_i : eigenvalue of $X^{-\frac{1}{2}}VX^{-\frac{1}{2}}$

g is concave in $t \Rightarrow f$ is concave

Convex function examples: norm, max, expectation

norm: If $f: R^n \rightarrow R$ is a norm and $0 \leq \theta \leq 1$

$$\begin{aligned} f(\theta x + (1 - \theta)y) &\leq f(\theta x) + f((1 - \theta)y) && \text{triangle inequality} \\ &= \theta f(x) + (1 - \theta)f(y) && \text{scalability} \end{aligned}$$

Max function: $f(x) = \max_i x_i, x = [x_1, x_2, \dots, x_n]^T$

$$\begin{aligned} f(\theta x + (1 - \theta)y) &= \max_i (\theta x_i + (1 - \theta)y_i) \\ &\leq \theta \max_i x_i + (1 - \theta) \max_i y_i \\ &= \theta f(x) + (1 - \theta)f(y) \quad \text{for } 0 \leq \theta \leq 1 \end{aligned}$$

Probability: (Expectation)

If $f(x)$ is convex with $p(x)$ a probability at x ,

$$\text{i. e. } p(x) \geq 0, \forall x \text{ and } \int p(x) dx = 1$$

Then $f(Ex) \leq Ef(x)$,

$$\text{where } Ex = \int x p(x) dx$$

$$Ef(x) = \int f(x) p(x) dx$$

1.3 Views of Functions and Related Hyperplanes

Given $f(x), x \in R^n$, we plot the function in R^n and R^{n+1} spaces.

1. Draw function in R^n space

Equipotential surface: **tangent plane** $\nabla f(\tilde{x})^T (x - \tilde{x}) = 0$ at \tilde{x}

2. Draw function in R^{n+1} space

2.1 Graph of function: $\{(x, h) | x \in \text{dom } f, h = f(x)\}$

hyperplane $(h = \nabla f(\tilde{x})^T (x - \tilde{x}) + f(\tilde{x}))$

$$[\nabla f(\tilde{x})^T \quad -1] \left(\begin{bmatrix} x \\ h \end{bmatrix} - \begin{bmatrix} \tilde{x} \\ f(\tilde{x}) \end{bmatrix} \right) = 0$$

Example: $f(x) = x^2$. We show the hyperplane with $\nabla f(x)$

2.2. Epigraph: $\text{epi } f: \{(x, t) | x \in \text{dom } f, f(x) \leq t\}$

A function is convex iff its epigraph is a convex set.

Example: $f(x) = \max\{f_i(x) | i = 1 \dots r\}$, $f_i(x)$ are convex.

Since $\text{epi } f$ is the intersect of $\text{epi } f_i$, $\text{epi } f$ is convex.

Thus, function f is convex.

2. Conditions of Optimality: First Order Condition

Definition: f is differentiable if $dom f$ is open and

$$\nabla f(x) \equiv \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \dots, \frac{\partial f(x)}{\partial x_n} \right)^T \text{ exists at each } x \in dom f$$

Theorem: Differentiable f with convex domain is convex

$$\text{iff } f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y \in dom f$$

Proof \Rightarrow If f is convex

$$\text{Then } (1 - t)f(x) + tf(y) \geq f((1 - t)x + ty), \forall 0 \leq t \leq 1$$

$$t[f(y) - f(x)] \geq f(x + t(y - x)) - f(x)$$

$$f(y) - f(x) \geq \frac{1}{t} (f(x + t(y - x)) - f(x))$$

$$= \nabla f(x)(y - x) \quad \text{when } t \rightarrow 0$$

$$\Leftarrow \text{Given } f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y \in dom f$$

$$\text{Let } z = (1 - t)x + ty$$

$$\text{where } \begin{cases} f(x) \geq f(z) + \nabla f(z)^T (x - z) \\ f(y) \geq f(z) + \nabla f(z)^T (y - z) \end{cases}$$

$$\text{Thus } (1 - t)f(x) + tf(y) \geq f(z)$$

2. Conditions: Second Order Condition

Definition: f is twice differentiable if $dom f$ is open and the Hessian $\nabla^2 f(x) \in S^n$

$$\nabla^2 f(x)_{ij} \equiv \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n \text{ exists at each } x \in dom f$$

Theorem: Twice Differentiable f with convex domain is convex
iff $\nabla^2 f(x) \succeq 0, \forall x \in dom f$

Proof: Using Lagrange remainder, we can find a z
 $f(x + t(y - x))$

$$= f(x) + \nabla f(x)^T t(y - x) + \frac{1}{2} t^2 (y - x)^T \nabla^2 f(z) (y - x),$$

$$\forall 0 \leq t \leq 1, z \text{ is between } x \text{ and } x + t(y - x)$$

Since the last term is always positive by assumption, the first order condition is satisfied.

2. Conditions: Second Order Condition

Example: Negative Entropy:

$$f(x) = x \log x, x \in R_{++}$$

$$f'(x) = \frac{x}{x} + \log x = 1 + \log x, f''(x) = \frac{1}{x}$$

Since $x \in R_{++}$, $f''(x) > 0 \Rightarrow f(x)$ is convex

Show the plot of $x \log x$

Remark:

- 1st order condition can be used to design and prove the property of opt. algorithms.
- 2nd order condition implies the 1st order condition
- 2nd order condition can be used to prove the convexity of the functions.

2. Conditions: Examples

- Quadratic Function: $f(x) = \frac{1}{2}x^T Px + q^T x + r, P \in S^n$
 $\nabla f(x) = Px + q, \nabla^2 f(x) = P$
- Least Square: $f(x) = \|Ax - b\|_2^2$
 $\nabla f(x) = 2A^T(Ax - b), \nabla^2 f(x) = A^T A$
- Quadratic over linear: $f(x, y) = \frac{x^2}{y}, y > 0$

$$\nabla f(x, y) = \left(\frac{2x}{y}, -\frac{x^2}{y^2} \right)^T,$$

$$\nabla^2 f(x, y) = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix} = \frac{2}{y^3} \begin{bmatrix} y & \\ & -x \end{bmatrix} \begin{bmatrix} y & \\ & -x \end{bmatrix}$$

2. Conditions: Examples

- Log-sum-exp: $f(x) = \log \sum_{k=1}^n e^{x_k}$ (Smooth max of softmax function)

$$\nabla^2 f(x) = \frac{1}{1^T z} \text{diag}(z) - \frac{1}{(1^T z)^2} z z^T, z_k = e^{x_k}$$

$$v^T \nabla^2 f(x) v = \frac{1}{(1^T z)^2} [(\sum_{i=1}^n z_i)(\sum_{i=1}^n v_i^2 z_i) - (\sum_{i=1}^n v_i z_i)^2] \geq 0,$$

for all $v \in R^n$ (Cauchy-Schwarz inequality)

Thus, $f(x)$ is a convex function

Cauchy-Schwarz inequality: $[(a^T a)(b^T b) \geq (a^T b)^2, a_i = \sqrt{z_i}, b_i = v_i \sqrt{z_i}]$

Proof 1: Let $z = a - \frac{a^T b}{b^T b} b$, or $a = z + \frac{a^T b}{b^T b} b$

We have

$$a^T a = z^T z + \frac{(a^T b)^2}{(b^T b)^2} b^T b \geq \frac{(a^T b)^2}{(b^T b)^2} b^T b = \frac{(a^T b)^2}{b^T b}$$

Proof 2: By induction

3. Operations that preserve convexity

- Nonnegative multiple: αf , where $\alpha \geq 0$, f is convex
- Sum: $f_1 + f_2$, where f_1 , and f_2 are convex
- Composition with affine function: $f(Ax + b)$, where f is convex

Proof: $\nabla_x^2 f(Ax + b) = A^T \nabla_y^2 f(y|y = Ax + b)A$

E.g. $f(x) = -\sum_{i=1}^m \log(b_i - a_i^T x_i)$,

$$\text{dom } f = \{x | a_i^T x < b_i, i = 1, \dots, m\}$$

$$f(x) = \|Ax + b\| \quad (\text{if } f \text{ is twice differentiable})$$

3. Operations that preserve convexity

- Pointwise maximum: $f(x) = \max\{f_1(x), \dots, f_r(x)\}$, f_i are convex
- Pointwise supremum:
$$g(x) = \sup_{y \in C} f(x, y),$$
 where $f(x, y)$ is convex in x and C is

an arbitrary set

Examples

- $S_C(x) = \sup_{y \in C} y^T x$, for an arbitrary set C
- $f(x) = \sup_{y \in C} \|x - y\|$, for an arbitrary set C
- $\lambda_{\max}(X) = \sup_{\|y\|_2=1} y^T X y$, $X \in S^n$

3. Operations that preserve convexity: Dual norm

Example:

$$f(x) = \max_{\|y\|_2 \leq 1} y^T x$$

$$f(x) = \max_{\|y\|_1 \leq 1} y^T x$$

$$f(x) = \max_{\|y\|_p \leq 1} y^T x$$

3. Operations that preserve convexity: max function

Theorem: Pointwise maximum of convex functions is convex

Given $f(x) = \max\{f_1(x), f_2(x)\}$, where f_1 and f_2 are convex and $\text{dom } f = \text{dom}\{f_1\} \cap \text{dom}\{f_2\}$ is convex, then $f(x)$ is convex.

Proof: For $0 \leq \theta \leq 1$, $x, y \in \text{dom } f$

$$f(\theta x + (1 - \theta)y)$$

$$= \max\{f_1(\theta x + (1 - \theta)y), f_2(\theta x + (1 - \theta)y)\}$$

$$\leq \max\{\theta f_1(x) + (1 - \theta)f_1(y), \theta f_2(x) + (1 - \theta)f_2(y)\}$$

$$\leq \theta \max\{f_1(x), f_2(x)\} + (1 - \theta) \max\{f_1(y), f_2(y)\}$$

$$= \theta f(x) + (1 - \theta)f(y)$$

i.e. $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$

Thus, function $f(x)$ is convex.

3. Operations that preserve convexity: minimization

Theorem: Partial minimization

If $g(x, y)$ is convex in x and y , and a set C is convex

Then $f(x) = \min_{y \in C} g(x, y)$ is convex.

Proof: Let $y_1 \in \{y \mid \min_{y \in C} g(x_1, y)\}$ and $y_2 \in \{y \mid \min_{y \in C} g(x_2, y)\}$,

we can write

$$\begin{aligned} & \theta f(x_1) + (1 - \theta)f(x_2) \\ &= \theta g(x_1, y_1) + (1 - \theta)g(x_2, y_2) \\ &\geq g(\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_2) \text{ (*g is convex*)} \\ &\geq \min_{y \in C} g(\theta x_1 + (1 - \theta)x_2, y) \text{ (*C is convex*)} \\ &= f(\theta x_1 + (1 - \theta)x_2) \end{aligned}$$

i.e. we have $\theta f(x_1) + (1 - \theta)f(x_2) \geq f(\theta x_1 + (1 - \theta)x_2)$

Therefore, $f(x) = \min_{y \in C} g(x, y)$ is convex.

3. Operations that preserve convexity

Examples for Partial Minimization

$$\text{Given } f(x, y) = [x^T \quad y^T] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x \in R^n, y \in R^m, A \in S_+^n, C \in S_+^m, \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in S_+^{n+m}$$

$$\text{Let } g(x) = \min_y f(x, y) = x^T (A - BC^+B^T)x,$$

C^+ : **pseudo inverse** of matrix C . (**Drazin inverse, or generalized inverse**)

We can claim that function $g(x)$ is convex.

Proof:

(1) $f(x, y)$ is convex

(2) $y \in R^m$ where R^m is a convex non-empty set

(3) Therefore, $g(x)$ is convex, i.e. $A - BC^+B^T \succcurlyeq 0$

3. Operations that preserve convexity

Composition:

Given $g: R^n \rightarrow R$ and $h: R \rightarrow R$, we set $f(x) = h(g(x))$

f is convex if g convex, h convex, \tilde{h} nondecreasing

g concave, h convex, \tilde{h} nonincreasing

f is concave if g convex, h concave, \tilde{h} nonincreasing

g concave, h concave, \tilde{h} nondecreasing

Proof : for $n=1$

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

Ex1: $\exp g(x)$ is convex if g is convex

Ex2: $1/g(x)$ is convex if g is concave and positive

Note that we set $\tilde{h}(x) = \infty$ if $x \notin \text{dom } h$, h is convex

$\tilde{h}(x) = -\infty$ if $x \notin \text{dom } h$, h is concave

3. Operations that preserve convexity

Show that $h(g(\theta x + (1 - \theta)y)) \leq \theta h(g(x)) + (1 - \theta)h(g(y))$

for the case that g, h are convex, and \tilde{h} is nondecreasing

(1) g is convex

$$g(\theta x + (1 - \theta)y) \leq \theta g(x) + (1 - \theta)g(y)$$

(2) h is nondecreasing: From (1), we have

$$h(g(\theta x + (1 - \theta)y)) \leq h(\theta g(x) + (1 - \theta)g(y))$$

(3) h is convex

$$h(\theta g(x) + (1 - \theta)g(y)) \leq \theta h(g(x)) + (1 - \theta)h(g(y))$$

(4) From (2) & (3)

$$h(g(\theta x + (1 - \theta)y)) \leq \theta h(g(x)) + (1 - \theta)h(g(y))$$

4. Conjugate Functions

The setting of conjugate functions starts from the following problem (which may not be convex)

$$\min f(x)$$

subject to

$$x \leq 0$$

We convert to a function of y

$$\inf_x f(x) - y^T x$$

The conjugate function is

$$f^*(y) = \sup_x y^T x - f(x)$$

In the class, we interchange min and inf; max and sup to simplify the notation.

4. Conjugate Functions

Given $f: R^n \rightarrow R$, we have $f^*: R^n \rightarrow R$

$$f^*(y) = \sup_{x \in \text{dom } f} y^T x - f(x); \quad (-f^*(y) = \min_{x \in \text{dom } f} -y^T x + f(x))$$

Constraint: $y \in R^n$ for which the supremum is finite (bounded)

$f^*(y)$ is called the conjugate of function f

Theorem : $f^*(y)$ is convex (pointwise maximum)

$$\text{Proof : } f^*(\theta y_1 + (1 - \theta)y_2) = \sup_x (\theta y_1 + (1 - \theta)y_2)^T x - f(x)$$

$$\begin{aligned} &\leq \sup_x \left(\theta y_1^T x - \theta f(x) \right) + \sup_x \left((1 - \theta) y_2^T x - (1 - \theta) f(x) \right) \\ &= \theta f^*(y_1) + (1 - \theta) f^*(y_2) \end{aligned}$$

Remark: $f^*(y)$ is convex even if $f(x)$ is not convex

4. Conjugate Functions

Suppose we have a pair \bar{x}, \bar{y} , such that $f^*(\bar{y}) = \bar{y}^T \bar{x} - f(\bar{x})$,
we can show that $\bar{y} = \nabla_x f(\bar{x})$ (exercise 3.40)

And the supporting hyperplane : $\bar{y}^T x - h = f^*(\bar{y})$

$$[\bar{y}^T \quad -1] \begin{bmatrix} x \\ h \end{bmatrix} = f^*(\bar{y})$$

Ex. $f(x) = x^2 - 2x, \quad x \in R$

$$f^*(y) = \sup_x yx - x^2 + 2x, \quad y \in R$$

4. Conjugate Functions

One way to view conjugate function

$$f^*(y) = \sup_{x \in \text{dom } f} y^T x - f(x)$$

x : negative slack

y : shadow price (loss) to accommodate the slack

$f^*(y)$: balance between price slack product ($y^T x$) and objective function $f(x)$.

Remark: When $f^*(y)$ is unbounded, the shadow price y is not reasonable.

4. Conjugate Functions: Examples (single variable)

Ex: $f(x) = ax + b, x \in R$

$$f^*(y) = \sup_x (yx - ax - b)$$

(1) If $y \neq a, f^*(y) = \infty$

(2) If $y = a, f^*(y) = -b \rightarrow \text{dom } f^* = a, f^*(y) = -b$

4. Conjugate Functions: Examples (single variable)

Ex: $f(x) = -\log x, \quad x \in \mathbb{R}_{++}$

$$f^*(y) = \sup_{x \in \mathbb{R}_{++}} yx + \log x$$

(1) If $y \geq 0, \quad f^*(y) = \infty$

(2) If $y < 0, \quad f^*(y) = \max_{x \in \mathbb{R}_{++}} xy + \log x$

Let $g(x) = xy + \log x, \quad g'(x) = y + \frac{1}{x}$

If $g'(x) = 0, \quad x = -\frac{1}{y}$

Thus, $f^*(y) = -1 + \log\left(-\frac{1}{y}\right) = -1 - \log(-y)$

$\rightarrow \text{dom } f^* = -\mathbb{R}_{++}, \quad f^*(y) = -1 - \log(-y)$

4. Conjugate Functions

Ex: $f(x) = e^x, x \in R$

$$f^*(y) = \sup_x xy - e^x$$

(1) $y < 0 : f^*(y) = \infty$

(2) $y > 0 : \text{Let } g(x) = xy - e^x \rightarrow g'(x) = y - e^x$

If $g'(x) = 0$, then $x = \log y$

Thus $f^*(y) = y \log y - y$

(3) $y = 0 : f^*(y) = 0 \rightarrow \text{dom } f^* = R_+, f^*(y) = y \log y - y$

Therefore, we have

$$f^*(y) = y \log y - y, \text{ where } y \geq 0.$$

4. Conjugate Functions

Ex: $f(x) = x \log x$, $x \in R_+$, $f(0) = 0$

$$f^*(y) = \sup_x xy - x \log x$$

Let $g(x) = xy - x \log x \rightarrow g'(x) = y - \log x - 1$

Suppose $g'(x) = 0$, we have $y = 1 + \log x$ or $x = e^{y-1}$

Thus $f^*(y) = ye^{y-1} - e^{y-1}(y-1) = e^{y-1}$ where $y \in R$

4. Conjugate Functions

Ex: $f(x) = \frac{1}{2}x^T Qx$, $x \in R^n$, $Q \in S_{+++}^n$

$$f^*(y) = \sup_x x^T y - \frac{1}{2}x^T Qx$$

Let $g(x) = x^T y - \frac{1}{2}x^T Qx \rightarrow \nabla g(x) = y - Qx$

If $\nabla g(x) = 0$, we have $x = Q^{-1}y$

Thus, $f^*(y) = \frac{1}{2}y^T Q^{-1}y$

Remark: Suppose that $f^*(\bar{y}) = \bar{y}^T \bar{x} - f(\bar{x})$ and $\nabla^2 f(\bar{x}) \succ 0$

We have $\nabla f^*(\bar{y}) = \bar{x}$ and $\nabla^2 f^*(\bar{y}) = (\nabla^2 f(\bar{x}))^{-1}$ (exercise 3.40)

4. Conjugate Functions

Basic Properties

(1) $f(x) + f^*(y) \geq x^T y$

Fenchel's inequality. Thus, in the above example

$$x^T y \leq \frac{1}{2} x^T Q x + \frac{1}{2} y^T Q^{-1} y, \quad \forall x, y \in R^n, Q \in S_{++}^n$$

(2) $f^{**} = f$, if f is convex & f is closed (i.e. $\text{epi } f$ is a closed set)

(3) If f is convex & differentiable, $\text{dom } f = R^n$

For $\max x^T y - f(x)$, we have $y = \nabla f(x^*)$

Thus, $f^*(y) = x^{*T} \nabla f(x^*) - f(x^*)$, $y = \nabla f(x^*)$

4. Conjugate Functions

$$\text{Ex : } f(x) = \log \sum_{i=1}^n e^{x_i} \leftrightarrow f^*(y) = \sum_{i=1}^n y_i \log y_i$$

$$f^*(y) = \sup_x y^T x - f(x) = \sup_x y^T x - \log \sum_{i=1}^n e^{x_i}$$

$$\text{Let } g(x) = y^T x - \log \sum_{i=1}^n e^{x_i}$$

$$\frac{\partial g(x)}{\partial x_i} = y_i - \frac{e^{x_i}}{\sum_{i=1}^n e^{x_i}} = 0$$

$$\text{Thus, } y_i = \frac{e^{x_i}}{\sum_{i=1}^n e^{x_i}}, \quad \text{i.e. } \mathbf{1}^T y = 1$$

(1) $\mathbf{1}^T y \neq 1 \rightarrow$ unbounded

(2) $y_i < 0 \rightarrow$ unbounded

(3) $f^*(y) = \sum_{i=1}^n y_i \log y_i, \quad y \geq 0, \mathbf{1}^T y = 1$

5. Log-Concave, Log-Convex Functions

Log function : $\log f(x)$, $f: R^n \rightarrow R, f(x) > 0, \forall x \in \text{dom } f$

Suppose f is twice differentiable, $\text{dom } f$ is convex.

$$\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^T$$

Then

f is log-convex iff $\forall x \in \text{dom } f$

$$f(x) \nabla^2 f(x) \geq \nabla f(x) \nabla f(x)^T$$

f is log-concave iff $\forall x \in \text{dom } f$

$$f(x) \nabla^2 f(x) \leq \nabla f(x) \nabla f(x)^T$$

5. Log-Concave, Log-Convex Functions

$$f : R^n \rightarrow R, \quad f(x) > 0, \forall x \in \text{dom } f$$

Definition: If $\log f$ is concave, f is log-concave.

Definition: If $\log f$ is convex, f is log-convex.

Ex : $f(x) = a^T x + b, \text{dom } f = \{x | a^T x + b\} : \text{log-concave}$

$$f(x) = x^a, \quad x \in R_{++}, \quad a \leq 0 : \text{log-convex}$$

$$a > 0 : \text{log-concave}$$

$$f(x) = e^{\alpha x} : \text{log convex \& log-concave}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du : \text{cumulative distribution function of}$$

Gaussian density log-concave

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})} : \text{log-concave}$$

5. Log-Concave, Log-Convex Functions

Properties

$$\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^T$$

$$f(x) \nabla^2 f(x) \geq \nabla f(x) \nabla f(x)^T, \quad \forall x \in \text{dom } f : \text{log-convex}$$

$$f(x) \nabla^2 f(x) \leq \nabla f(x) \nabla f(x)^T, \quad \forall x \in \text{dom } f : \text{log-concave}$$

Outlines

1. Definitions: Convexity, Examples & Views
2. Conditions of Optimality
 1. First Order Condition
 2. Second Order Condition
3. Operations that Preserve the Convexity
 1. Pointwise Maximum
 2. Partial Minimization
4. Conjugate Function
5. Log-Concave, Log-Convex Functions