1.3 Views of Functions and Related Hyperplanes

Given $f(x), x \in R^n$, we plot the function in R^n and R^{n+1} spaces.

1. Draw function in R^n space

   Equipotential surface: **tangent plane** $\nabla f(\tilde{x})^T(x-\tilde{x}) = 0$ at $\tilde{x}$

2. Draw function in R^{n+1} space

   2.1 Graph of function: $\{(x,h) | x \in dom\, f, h = f(x)\}$

   hyperplane ($\mathrm{h} = \nabla f(\tilde{x})^T(x-\tilde{x}) + f(\tilde{x})$)

   $$[\nabla f(\tilde{x})^T \quad -1]\left(\begin{bmatrix} x \\ h \end{bmatrix} - \begin{bmatrix} \tilde{x} \\ f(\tilde{x}) \end{bmatrix}\right) = 0$$ supporting hyperplane

   Example: $f(x) = x^2$. We show the hyperplane with $\nabla f(x)$

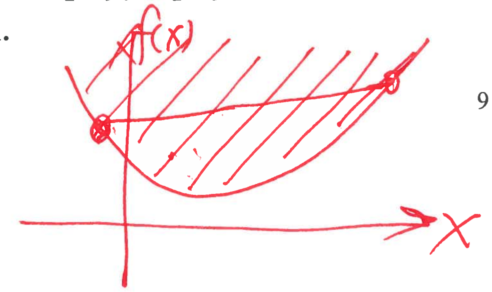
   2.2. Epigraph: epi f: $\{(\mathrm{x}, t) | x \in dom\, f, f(x) \le t\}$

   A function is convex iff its epigraph is a convex set.

   Example: $f(x) = \max\{f_i(x) |\ i = 1 \ldots r\}, f_i(x)\ are\ convex.$

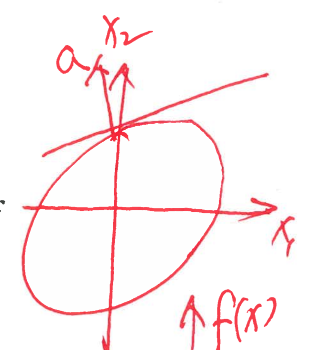
   Since epi f is the intersect of epi f_i, epi f is convex.

   Thus, function f is convex.



9

2. Conditions of Optimality: First Order Condition

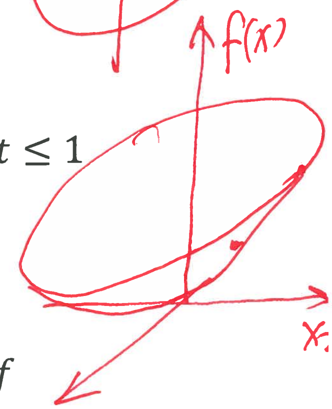


Defintion: f is differentiable if $dom f$ is open and

$$\nabla f(x) \equiv \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n}\right)^T \text{ exists at each } x \in dom f$$

Theorem: Differentiable f with convex domain is convex

$$\text{iff } f(y) \ge f(x) + \nabla f(x)^{\mathrm{T}}(y-x), \forall x, y \in dom f$$



Proof => If f is convex

$$Then\ (1-t)f(x) + tf(y) \ge f((1-t)x + ty), \forall 0 \le t \le 1$$

$$t[f(y) - f(x)] \ge f(x + t(y-x)) - f(x)$$

$$f(y) - f(x) \ge \frac{1}{t}(f(x+t(y-x)) - f(x))$$

$$= \nabla f(x)(y-x) \quad when\ t \to 0$$

<= $Given\ f(y) \ge f(x) + \nabla f(x)^{\mathrm{T}}(y-x), \forall x, y \in dom f$

$Let\ z = (1-t)x + ty$

$$where \begin{cases} f(x) \ge f(z) + \nabla f(z)^{\mathrm{T}}(x-z) & (1-t)f(z) + (1-t)\nabla f(z)^T(x-z) \\ f(y) \ge f(z) + \nabla f(z)^{\mathrm{T}}(y-z) & tf(z) + t\nabla f(z)^T(y-z) \end{cases}$$

$Thus\ (1-t)f(x) + tf(y) \ge f(z)$

10

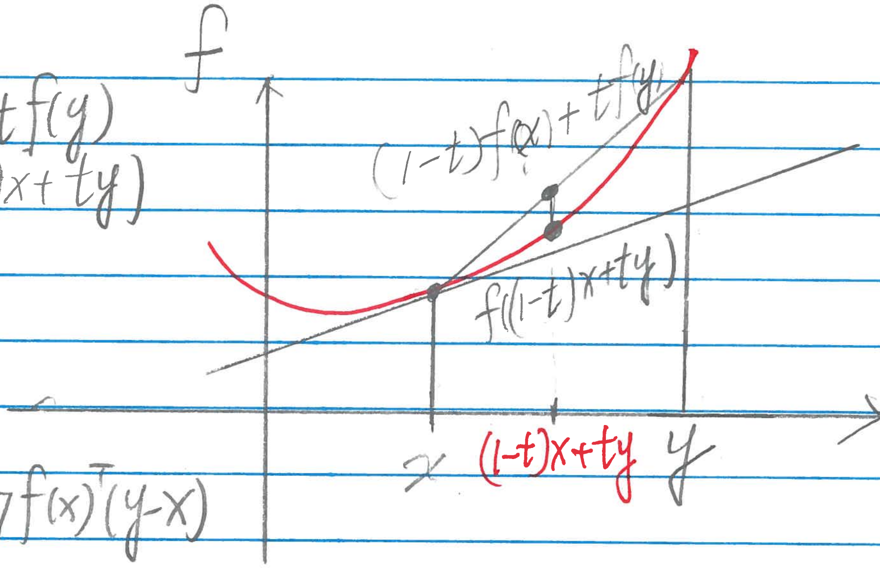
W3B

Supporting Hyperplane $[\nabla f(x)^T \ -1]\left(\begin{bmatrix} y \\ f(y) \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix}\right) \le 0$

$(1-t)x - (1-t)z + y - tz$

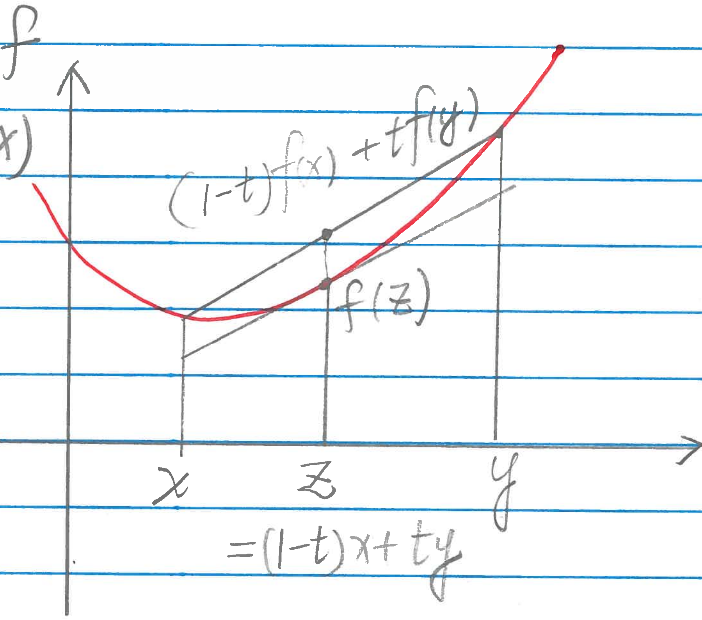
First Order Condition

If $(1-t)f(x)+tf(y)$
$\geq f((1-t)x+ty)$



Then
$f(y) \geq f(x)+\nabla f(x)^T(y-x)$

If $f(y) \geq f(x)+\nabla f(x)^T(y-x)$



Then $(1-t)f(x)+tf(y)$
$\geq f((1-t)x+ty)$

① $f(x) \geq f(z)+\nabla f(z)^T(x-z)$

② $f(y) \geq f(z)+\nabla f(z)^T(y-z)$

2. Conditions: Second Order Condition

Definition: f is twice differentiable if $dom f$ is open and the Hessian $\nabla^2 f(x) \in S^n$

$$\nabla^2 f(x)_{ij} \equiv \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i,j = 1, \dots, n \text{ exists at each } x \in dom f$$

Theorem: Twice Differentiable f with convex domain is convex iff $\nabla^2 f(x) \succcurlyeq 0, \forall x \in dom f$

Proof: Using Lagrange remainder, we can find a z

$$f(x + t(y - x))$$
$$= f(x) + \nabla f(x)^T t(y - x) + \frac{1}{2} t^2 (y - x)^T \nabla^2 f(z)(y - x),$$
$$\forall 0 \le t \le 1, z \text{ is between } x \text{ and } x + t(y - x)$$

Since the last term is always positive by assumption, the first order condition is satisfied.

2. Conditions: Second Order Condition

Example: Negative Entropy: $\sum_i - p_i \ln p_i$

☆ $f(x) = x \log x, x \in R_{++}$

$$f'(x) = \frac{x}{x} + \log x = 1 + \log x, f''(x) = \frac{1}{x}$$

Since $x \in R_{++}, f''(x) > 0 \Rightarrow f(x)$ is convex

Show the plot of $x \log x$

Remark:

- 1st order condition can be used to design and prove the property of opt. algorithms.
- 2nd order condition implies the 1st order condition
- 2nd order condition can be used to prove the convexity of the functions.

2. Conditions: Examples

- Quadratic Function: $f(x) = \frac{1}{2}x^T Px + q^T x + r, P \in S^n$
  $$\nabla f(x) = Px + q,\ \nabla^2 f(x) = P$$
- Least Square: $f(x) = \|Ax - b\|_2^2$
  $$\nabla f(x) = 2A^T(Ax - b),\ \nabla^2 f(x) = A^T A$$
- Quadratic over linear: $f(x,y) = \frac{x^2}{y},\ y > 0$

$$\nabla f(x,y) = \left(\frac{2x}{y}, -\frac{x^2}{y^2}\right)^T,$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix} = \frac{2}{y^3}\begin{bmatrix} y \\ -x \end{bmatrix}\begin{bmatrix} y & -x \end{bmatrix}$$

Positive Semidefinite

$X \succeq 0$ ① $X = X^T$ $X \in R^{n\times n}$

② $y^T X y \geq 0\ \forall y \in R^n$.

$y^T X y > 0 \Rightarrow X \succ 0$.

$X = D\Sigma D^T$ $DD^T = I$

$\Sigma = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

13

2. Conditions: Examples

- Log-sum-exp: $f(x) = \log \sum_{K=1}^{n} e^{x_k}$ (Smooth max of softmax function)
  $\nabla^2 f(x) = \frac{1}{1^T z} diag(z) - \frac{1}{(1^T z)^2} zz^T, z_k = e^{x_k}$
  $v^T \nabla^2 f(x) v = \frac{1}{(1^T z)^2}[(\sum_{i=1}^{n} z_i)(\sum_{i=1}^{n} v_i^2 z_i) - (\sum_{i=1}^{n} v_i z_i)^2] \geq 0,$
  for all $v \in R^n$ (Cauchy-Schwarz inequality)
  Thus, $f(x)$ is a convex function

$\sum_i a_i^2 \sum_i b_i^2 \geq (\sum_i a_i b_i)^2$

Cauchy-Schwarz inequality: $[(a^T a)(b^T b) \geq (a^T b)^2,\ a_i = \sqrt{z_i}, b_i = v_i\sqrt{z_i}]$

Proof 1: Let $z = a - \frac{a^T b}{b^T b} b$, or $a = z + \frac{a^T b}{b^T b} b$

We have

$$a^T a = z^T z + \frac{(a^T b)^2}{(b^T b)^2} b^T b \geq \frac{(a^T b)^2}{(b^T b)^2} b^T b = \frac{(a^T b)^2}{b^T b}$$

Proof 2: By induction

3. Operations that preserve convexity

- Nonnegative multiple: αf, where $\alpha \geq 0$, f is convex
- Sum: $f_1 + f_2$, where f_1, *and* f_2 are convex
- Composition with affine function: $f(Ax+b)$, where f is convex $\{f(y) \mid y = Ax+b\}$

  Proof: $\nabla_x^2 f(Ax+b) = A^T \nabla_y^2 f(y|y = Ax+b)A$

  E.g. $f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x_i)$,

  $dom\, f = \{x | a_i^T x < b_i, i = 1, \ldots, m\}$

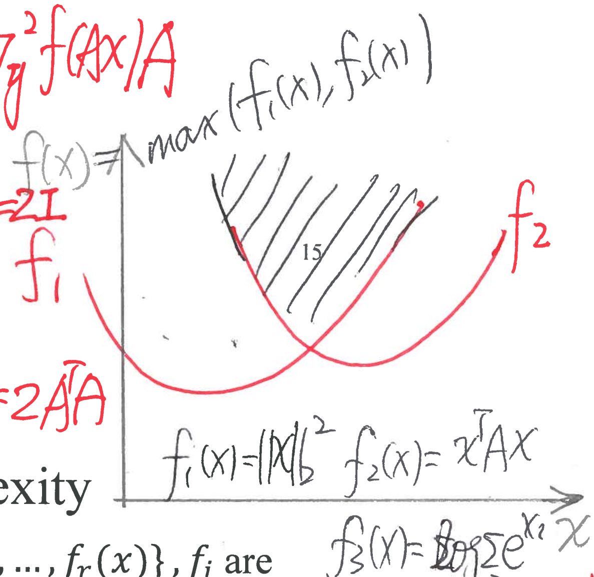
  $f(x) = \|Ax+b\|$ (if f is twice differentiable)

$\nabla_y f(y) \Rightarrow A^T \nabla_y f(Ax) \Rightarrow A^T \nabla_y^2 f(Ax) A$

$f(y) = y^T y \Rightarrow \nabla_y f(y) = 2y \quad \nabla_y^2 f(y) = 2I$

$f(Ax) = x^T A^T A x$

$\nabla_x f(Ax) = 2A^T A x \quad \nabla_x^2 f(Ax) = 2A^T A$



3. Operations that preserve convexity

- Pointwise maximum: $f(x) = \max\{f_1(x), \ldots, f_r(x)\}$, f_i are convex

  f_i convex function $\rightarrow$ epigraph f_i convex $\rightarrow \bigcap_i$ epigraph f_i convex $\rightarrow \max_i f_i$ convex
- Pointwise supremum:

  $g(x) = \sup_{y \in C} f(x,y)$, where $f(x,y)$ is convex in x and C is an arbitrary set

  Examples
  - $S_c(x) = \sup_{y \in C} y^T x$, for an arbitrary set C
  - $f(x) = \sup_{y \in C} \|x - y\|$, for an arbitrary set C
  - $\lambda_{max}(X) = \sup_{\|y\|_2 = 1} y^T X y$, $X \in S^n$

$y^T X y = \sum_i \sum_j X_{ij} y_i y_j$ linear function of X_{ij}

3. Operations that preserve convexity: Dual norm

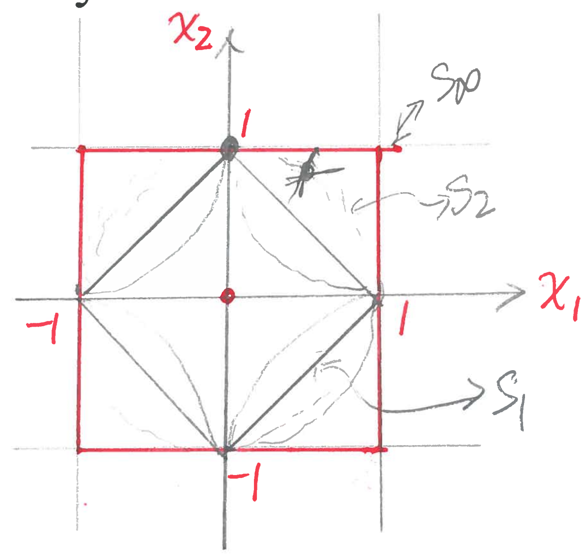
Example:

$f(x) = \max_{\|y\|_2 \le 1} y^T x = \|x\|_2$

$f(x) = \max_{\|y\|_1 \le 1} y^T x = \|x\|_\infty$

$f(x) = \max_{\|y\|_p \le 1} y^T x = \|x\|_q$

$\frac{1}{q} + \frac{1}{p} = 1$ or $q = \frac{p}{p-1}$



$\underbrace{\{x \mid \|x\|_\infty = 1\}}_{S_\infty}$ $\underbrace{\{x \mid \|x\|_2 = 1\}}_{S_2}$ $\underbrace{\{x \mid \|x\|_1 = 1\}}_{S_1}$

$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$

3. Operations that preserve convexity: max function

Theorem: Pointwise maximum of convex functions is convex
Given $f(x) = \max\{f_1(x), f_2(x)\}$, where f_1 and f_2 are convex and $dom\ f = dom\{f_1\} \cap dom\{f_2\}$ is convex, then $f(x)$ is convex.

Proof: For $0 \le \theta \le 1$, $x, y \in dom\ f$

$$
\begin{aligned}
&f(\theta x + (1-\theta)y) \\
&\quad = \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\
&\quad \le \max\{\theta f_1(x) + (1-\theta)f_1(y), \theta f_2(x) + (1-\theta)f_2(y)\} \\
&\quad \le \theta \max\{f_1(x), f_2(x)\} + (1-\theta)\max\{f_1(y), f_2(y)\} \\
&\quad = \theta f(x) + (1-\theta)f(y)
\end{aligned}
$$

i.e. $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y)$

Thus, function $f(x)$ is convex.

3. Operations that preserve convexity: minimization

Theorem: Partial minimization

If $g(x, y)$ is convex in x and y, and a set C is convex

Then $f(x) = \min_{y \in C} g(x, y)$ is convex.

Proof: Let $y_1 \in \{y | \min_{y \in C} g(x_1, y)\}$ and $y_2 \in \{y | \min_{y \in C}(g(x_2, y)\}$,

we can write

$$\begin{aligned}
&\theta f(x_1) + (1-\theta) f(x_2) \\
&\quad = \theta g(x_1, y_1) + (1-\theta) g(x_2, y_2) \\
&\quad \geq g(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta) y_2) \ (g \ is \ convex) \\
&\quad \geq \min_{y \in C} g(\theta x_1 + (1-\theta)x_2, y) \ (C \ is \ convex) \\
&\quad = f(\theta x_1 + (1-\theta)x_2)
\end{aligned}$$

i.e. we have $\theta f(x_1) + (1-\theta) f(x_2) \geq f(\theta x_1 + (1-\theta)x_2)$

Therefore, $f(x) = \min_{y \in C} g(x, y)$ is convex.

3. Operations that preserve convexity

Examples for Partial Minimization

Given $f(x, y) = \begin{bmatrix} x^T & y^T \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$x \in R^n, y \in R^m, A \in S_+^n, C \in S_+^m, \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in S_+^{n+m}$$

Let $g(x) = \min_y f(x, y) = x^T(A - BC^+B^T)x$,

C^+: **pseudo inverse** of matrix C. (Drazin inverse, or generalized inverse)

We can claim that function $g(x)$ is convex.

Proof:

(1) $f(x, y)$ is convex

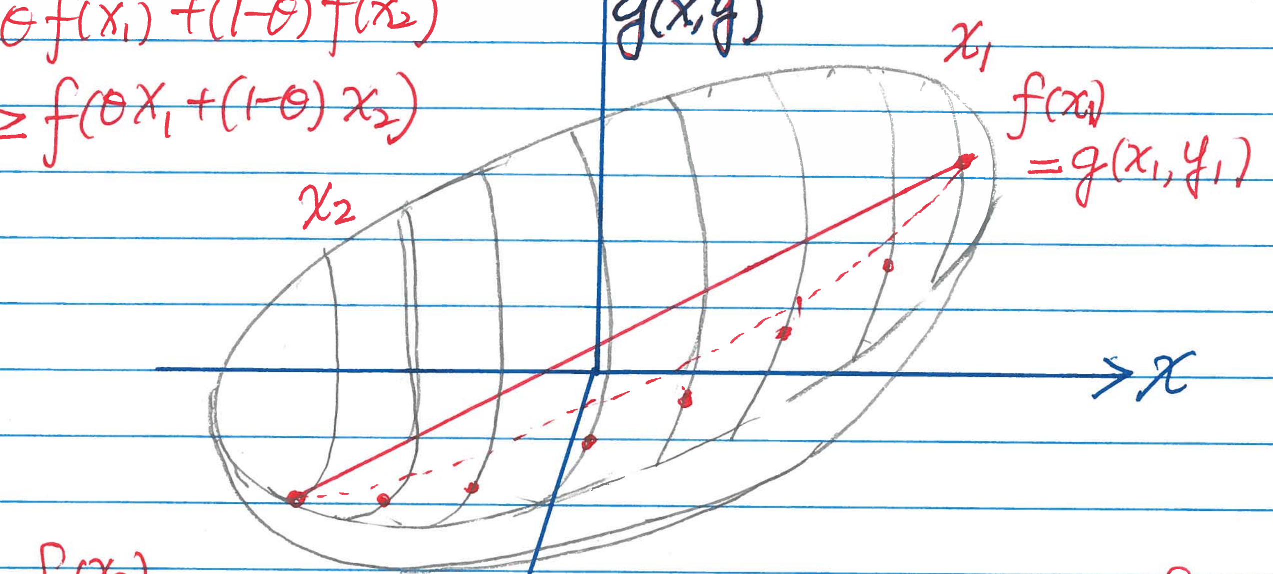
(2) $y \in R^m$ where R^m is a convex non-empty set

(3) Therefore, $g(x)$ is convex, i.e. $A - BC^+B^T \succcurlyeq 0$

Show $f(x) = \min_{y \in C} g(x,y)$, g is convex in x & y & C is convex.

$\theta f(x_1) + (1-\theta) f(x_2)$

$\geq f(\theta x_1 + (1-\theta) x_2)$



(1). $\theta f(x_1) + (1-\theta) f(x_2)$

$= \theta g(x_1, y_1) + (1-\theta) g(x_2, y_2)$

$\geq g(\theta x_1 + (1-\theta) x_2, \theta y_1 + (1-\theta) y_2)$

(2) $\geq \min_{y \in C} g(\theta x_1 + (1-\theta) x_2, y)$

$= f(\theta x_1 + (1-\theta) x_2)$