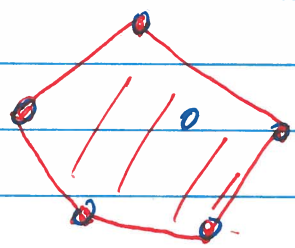
Convex Hull:

A convex hull of a set C is the smallest convex set that contains C.



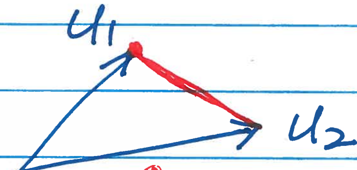
$$\left\{ \sum_i \theta_i u_i \;\middle|\; u_i \in C, \sum \theta_i = 1, \theta_i \geq 0 \ \forall i \right\}$$

$\theta_1 = 1 - \theta_2 \qquad 0 \leq \theta_2 \leq 1$

$(1-\theta_2)$

I. $\{\theta_1 u_1 + \theta_2 u_2 \mid \underline{\theta_1 + \theta_2 = 1}, \theta_1, \theta_2 \geq 0\}$

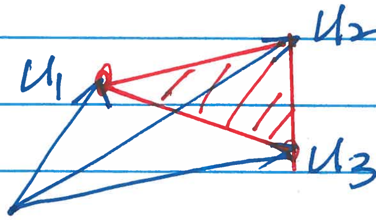
$\{u_1 + \theta_2 (u_2 - u_1) \mid 1 \geq \theta_2 \geq 0\}$



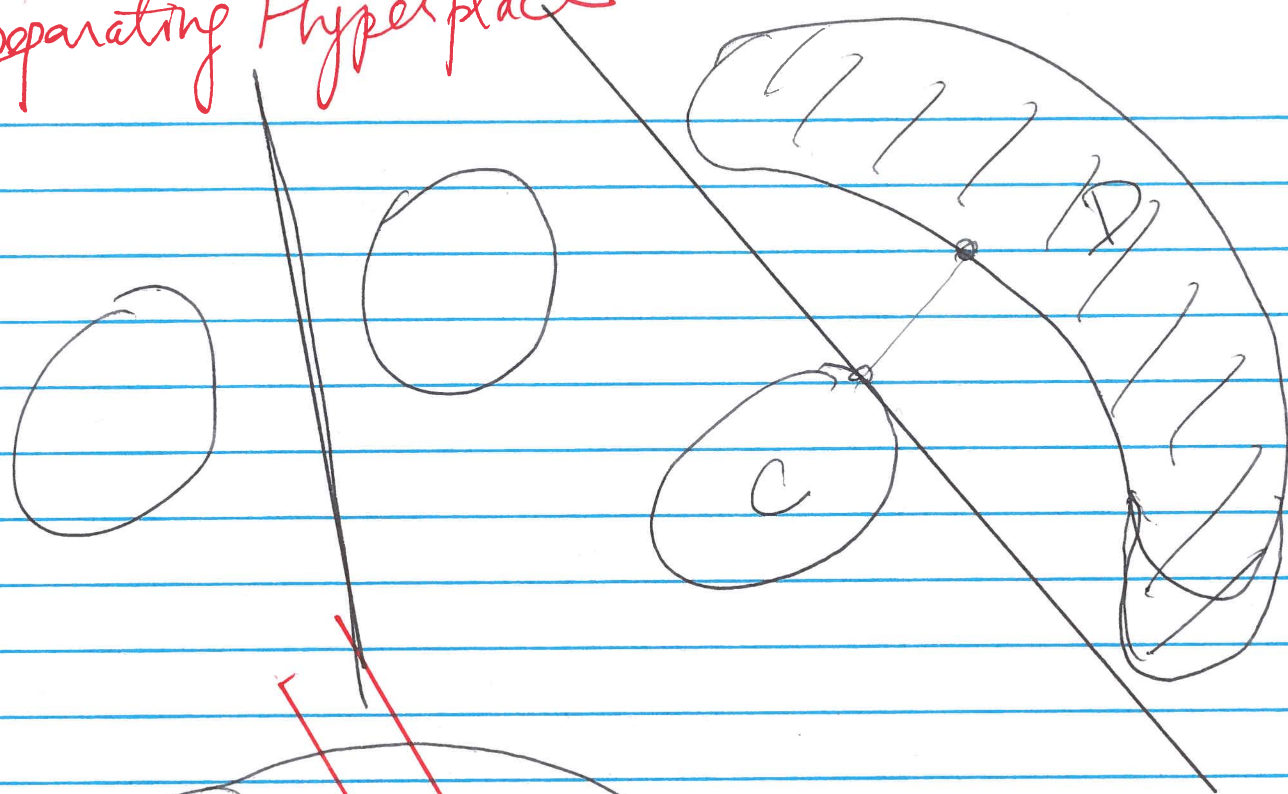
$1-\theta_2-\theta_3 \qquad \theta_1 = 1 - \theta_2 - \theta_3 \geq 0$

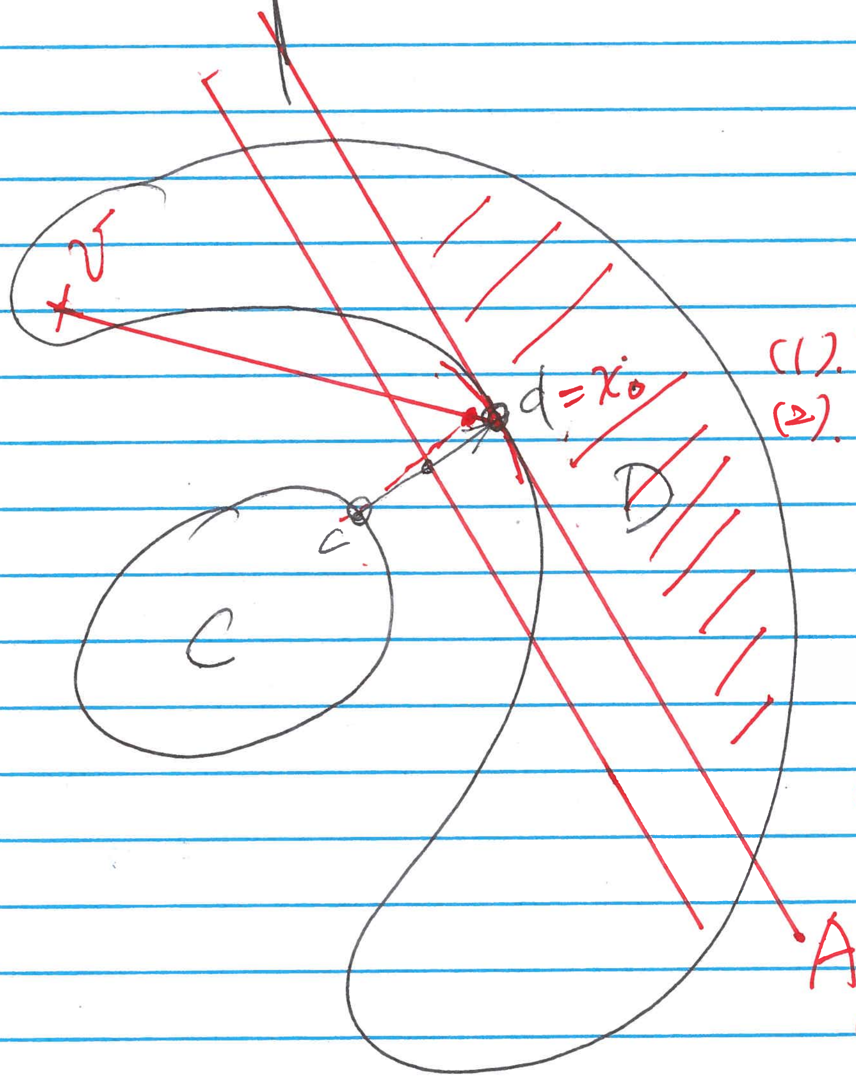
II $\{\theta_1 u_1 + \theta_2 u_2 + \theta_3 u_3 \mid \underline{\theta_1 + \theta_2 + \theta_3 = 1}, \theta_1, \theta_2, \theta_3 \geq 0\}$

$\{u_1 + \theta_2 (u_2 - u_1) + \theta_3 (u_3 - u_1) \mid \underline{\theta_2 + \theta_3 \leq 1}, \theta_2, \theta_3 \geq 0\}$



Separating Hyperplace





(1). D is convex

(2). dist(c, d) = dist(C, D)

~~$a^T x = a$~~

$a^T(x-d) \geq 0$ $\forall x \in D$

$a^T(v-d) < 0$

Cone K | Dual Cone K^*

$\left\{ \sum_{i=1}^{k} \theta_i u_i \;\middle|\; \theta_i \geq 0, \forall i \right\}$ $\quad$ $\left\{ y \;\middle|\; x^T y \geq 0, \forall x \in K \right\}$

$u_i \in R^n$

(1) $x, y \in R^n$

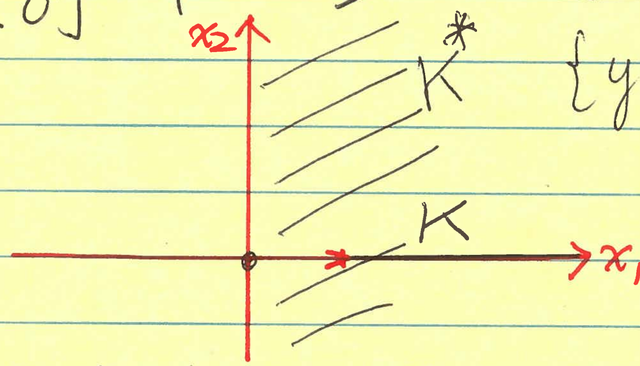
$y^T(\sum \theta_i u_i) \geq 0 \;\; \forall \theta_i \geq 0.$

(2)

$\left\{ y \;\middle|\; u_i^T y \geq 0, \forall u_i \in K \right\}$

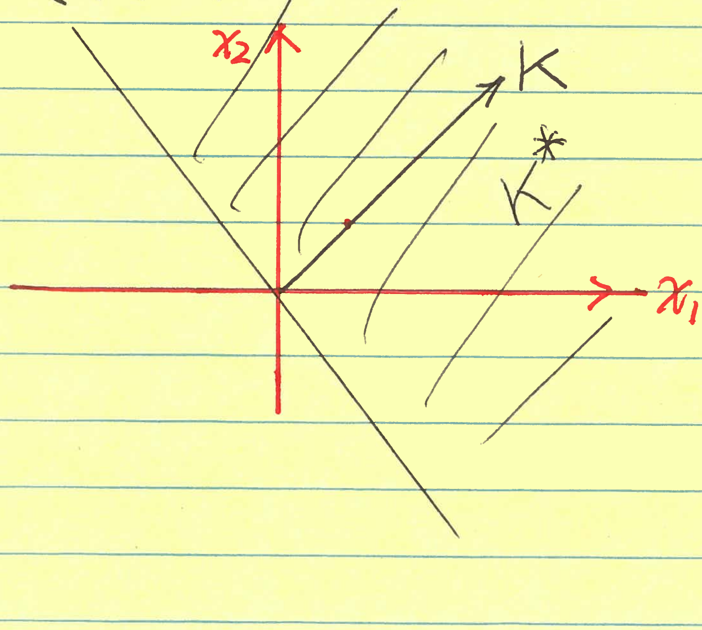
Ex $\quad \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \;\middle|\; \theta \geq 0 \right\}$ $\quad$ $\left\{ y \;\middle|\; \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq 0 \right\}$

$\left\{ y \;\middle|\; y_1 \geq 0 \right\}$



Ex $\quad \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \theta \;\middle|\; \theta \geq 0 \right\}$ $\quad$ $\left\{ y \;\middle|\; \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq 0 \right\}$

$y_1 + y_2 \geq 0$

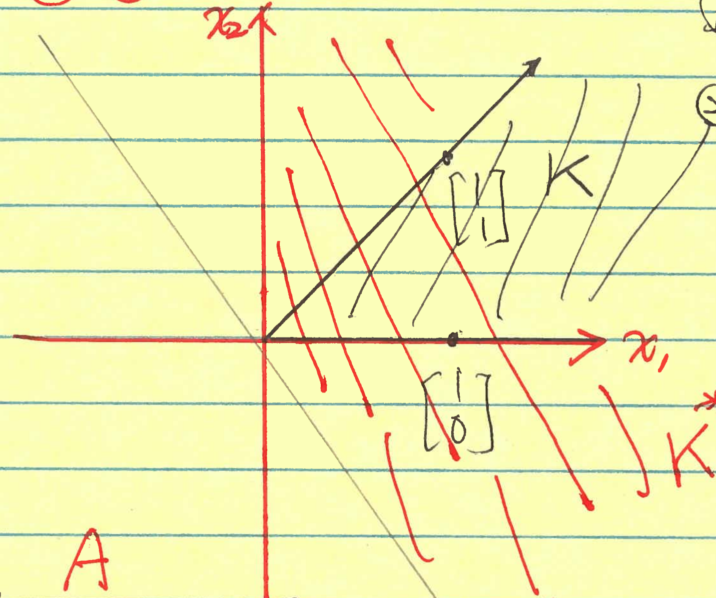


Ex

$$\left\{ \overbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}}^{A}\begin{bmatrix}\theta_1\\\theta_2\end{bmatrix} \middle| \theta_1,\theta_2\ge 0\right\}^{K} \quad \left\{ y \middle| \overbrace{\begin{bmatrix}1&0\\1&1\end{bmatrix}}^{A^T} y \ge \begin{bmatrix}0\\0\end{bmatrix}\right\}^{K^*}$$

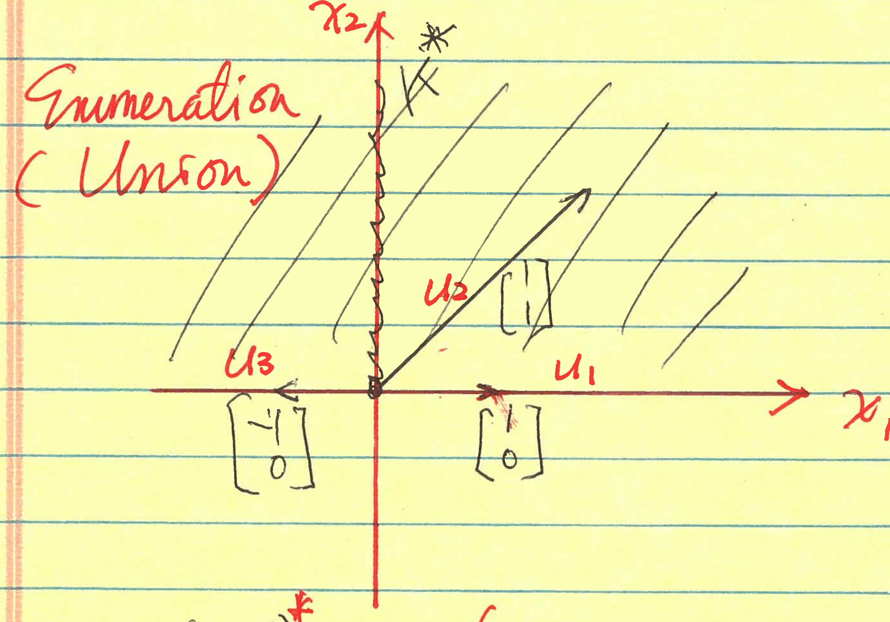
① $y_1\ge 0$

② $y_1+y_2\ge 0.$



Ex

$$\left\{ \overbrace{\begin{bmatrix}1&1&-1\\0&1&0\end{bmatrix}}^{A}\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\end{bmatrix} \middle| \theta_1,\theta_2,\theta_3\ge 0\right\} \quad \left\{ y \middle| \overbrace{\begin{bmatrix}1&0\\1&1\\-1&0\end{bmatrix}}^{A^T} y \ge \begin{bmatrix}0\\0\end{bmatrix}\right\}$$



Qualification
(Intersect)

$y_1\ge 0.$

$y_1+y_2\ge 0$

$-y_1\ge 0$

$\Rightarrow y_1=0$

$(K^*)^* = K$ (conditions)

3. Separating Hyperplane

$\{x|a^T x = b\}$ (Classification, Optimization, Duality)

Theorem : Given two convex sets $C \cap D = \emptyset$ in R^n

$$\exists a \in R^n, b \in R, \quad s.t.\ a^T x \le b, \forall x \in C$$

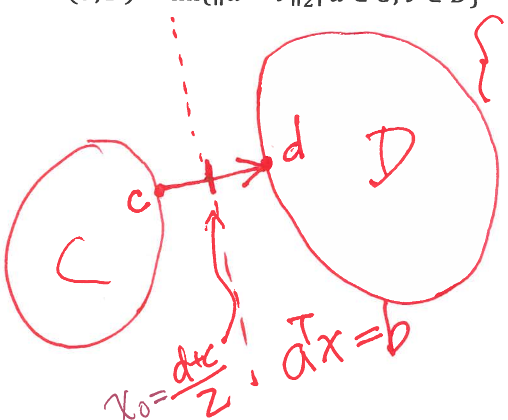
$$a^T x \ge b, \forall x \in D$$

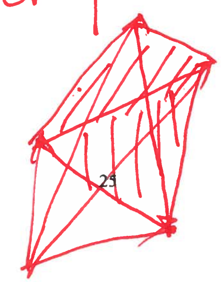
Actually, $a = d - c, b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}$ x_0 $\frac{(d-c)^T(d+c)}{2}$

i.e. $f(x) = a^T x - b = (d-c)^T\left(x - \frac{d+c}{2}\right)$

For $dist(C, D) = \inf\{\|u - v\|_2 | u \in C, v \in D\}$

$\{U\theta | 1^T\theta = 1, \theta \ge 0\}$





3. Separating Hyperplane

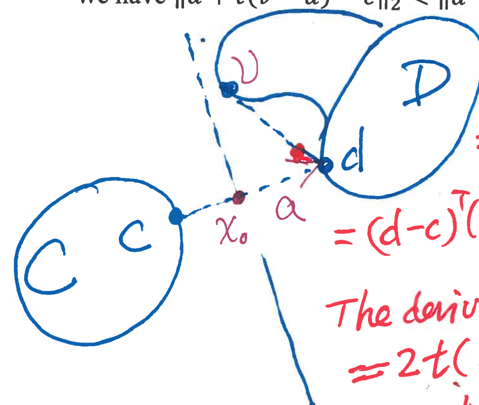
Proof : $\forall v \in D, a^T v \ge a^T d$ should be true if $dist(C,D) = dist(c,d)$

By contradiction, suppose that $a^T v < a^T d$

Then we can show that $d + t(v - d)$ is close to c for $t > 0$

Because $\frac{d}{dt}\|d + t(v-d) - c\|_2^2 = 2(d-c)^T(v-d) < 0$ $(t \to 0)$

We have $\|d + t(v - d) - c\|_2 < \|d - c\|_2$ for tiny $t > 0$



$\|d + t(v-d) - c\|_2^2$

$= \|d - c + t(v-d)\|_2^2$

$= (d-c)^T(d-c) + t^2(v-d)^T(v-d)$
$+ 2t(d-c)^T(v-d)$

The derivative

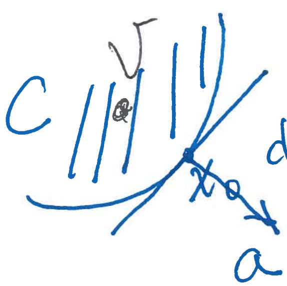
$= 2t(v-d)^T(v-d)$
$+ 2(d-c)^T(v-d)$

3. Supporting Hyperplane

Given set $C \in R^n$, and a point x_0 on the boundary of C, the hyperplan $\{x|a^T x = a^T x_0\}$ is called supporting hyperplane of C if $a^T x \le a^T x_0, \forall x \in C$.

Supporting Hyperplane Theorem: For any nonempty convex set C, and a point x_0 on the boundary of C, There exists a support hyperplane to C at x_0.

Proof: A separating hyperplane that separates interior C and $\{x_0\}$ is a supporting hyperplane.



direction that obj decreases

$a^T x_0 = b$

$a^T v \le b$

4. Dual Cones

Given Cone K (i.e. $K = \{\sum_{i=1}^k \theta_i u_i | \theta_i \ge 0, u_i \in R^n, \forall i\}$)

$K^* = \{y | x^T y \ge 0, \forall x \in K\}$

Ex: 1. $K = R_+^n : K^* = R_+^n$

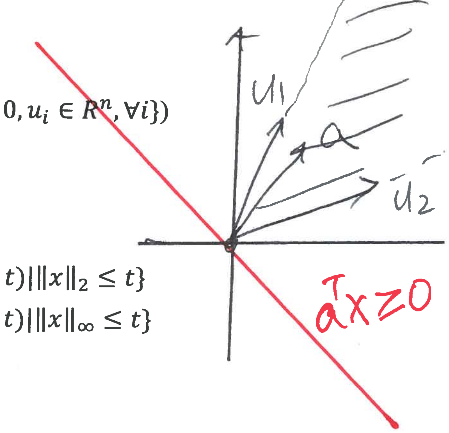
2. $K = S_+^n : K^* = S_+^n$

3. $K = \{(x,t) | \|x\|_2 \le t\} : K^* = \{(x,t) | \|x\|_2 \le t\}$

4. $K = \{(x,t) | \|x\|_1 \le t\} : K^* = \{(x,t) | \|x\|_\infty \le t\}$

Examples

W2B



4. Dual Cones

Show that cone $K = \{(x,t) \mid \|x\|_1 \le t\}$ has its dual

$$K^* = \{(x,t) \mid \|x\|_\infty \le t\}$$

Proof :

Statement $x^T u + tv \ge 0, \forall \|x\|_1 \le t \leftrightarrow \|u\|_\infty \le v$

L=>R By contradiction, suppose that $\|u\|_\infty > v$

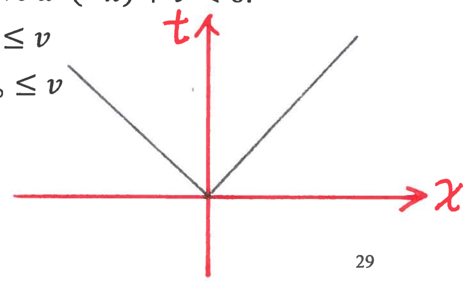
We can find $\exists x\ s.t\ \|x\|_1 \le 1, x^T u > v$

Setting t=1, then we have $u^T(-x) + v < 0.$

R=>L Given $\|x\|_1 \le t, \|u\|_\infty \le v$

$u^T \|-x/t\|_1 \le \|u\|_\infty \le v$

Thus, $u^T(-x) \le vt$



4. Dual Cones

Definition: $x \preceq_K y$ if $y - x \in K$

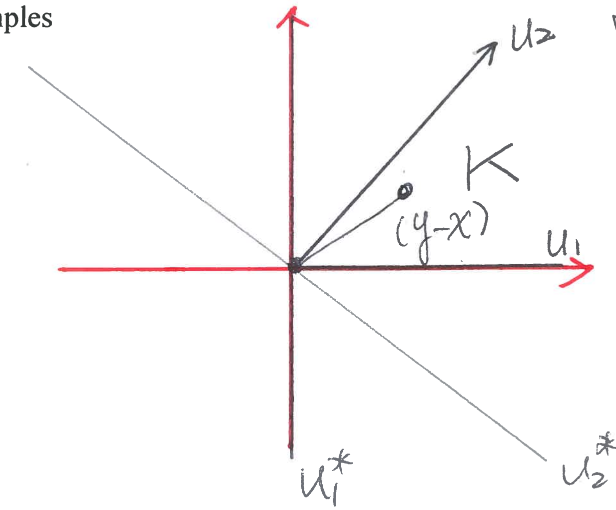
Theorem: $x \preceq_K y$ iff $\lambda^T x \le \lambda^T y, \forall \lambda \in K^*$

Examples

$\min a^T X$

$f(x) = a^T X$

$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f(}{\partial x_n} \end{bmatrix}$



4. Dual Cones

The polyhedral cone $V = \{x \mid Ax \ge 0\}$ has its dual cone

$V^* = \{A^T v \mid v \ge 0\}$

Proof : By definition

$V^* = \{y \mid x^T y \ge 0, \forall x \in V\}$

Thus $V^* = \{y \mid x^T y \ge 0, \forall Ax \ge 0\}$ AX

Let $y = A^T v$, we have $x^T y = \underline{x^T A^T v} > 0$ if $v \ge 0$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ i.e. $x_1 + 2x_2 \ge 0, x_1 - x_2 \ge 0$

$A^T = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ i.e. $\{\theta_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \theta_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mid \theta_1, \theta_2 \ge 0\}$

4. Dual Cones

Remark: $\{x_0 + \Delta x \mid \Delta x \in K\}$

(1) K cone can be translated to x_0

(2) If $a \in K^*$, then $a^T x \ge 0, \forall x \in K$, i.e. $-ax$ is a supporting hyperplane of cone K

(3) Suppose that the current feasible search region is at corner x_0 and $\{x_0 + \Delta x \mid \Delta x \in K, \|\Delta x\| < r\}$ is a local feasible region of a convex set

If $\nabla f_0(x_0) \in K^*$, i.e. $\nabla f_0(x_0)^T \Delta x \ge 0, \forall \Delta x \in K$,

Then x_0 is an optimal solution