CSE203B Convex Optimization
Lecture 2 Convex Set

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Convex Optimization Problem:

$f_i(x): R^n \to R$

Gradiant descent
Newton iterations

$\min_x f_0(x),\ x \in R^n$

Subject to limit the search space

$f_i(x) \le b_i\ , i = 1, \cdots, m$

n, m can be very large

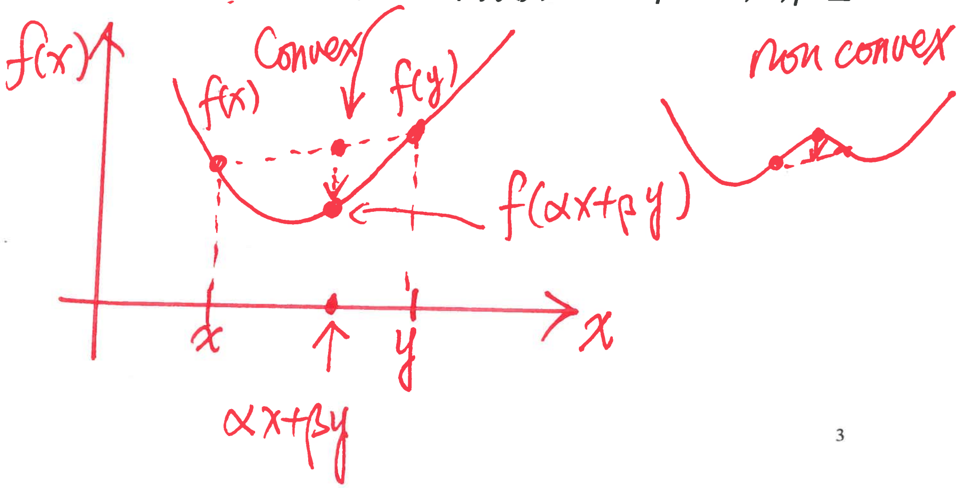
1. $f_0(x)$ is a convex function
2. For $f_i(x) \le b_i, i = 1, \dots, m$

   $\{x | f_i(x) \le b_i, i = 1, \cdots, m\}$ is a convex set

Convex Optimization Problem:

A. Convex Function Definition:

$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y), \forall \alpha + \beta = 1, \alpha, \beta \ge 0$



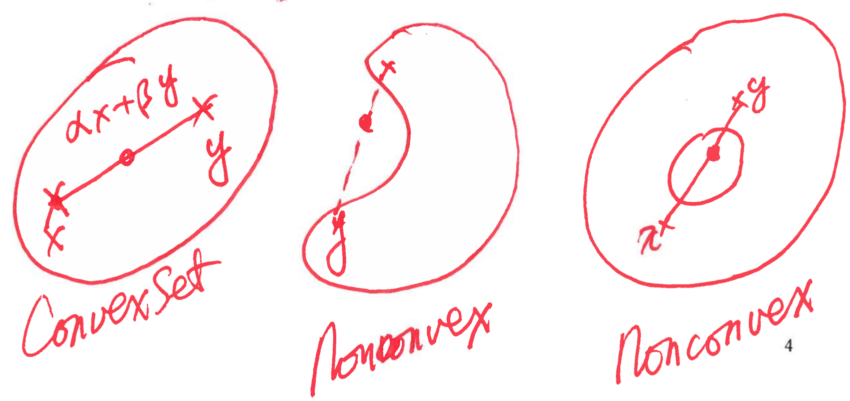
Convex Optimization Problem:

A. Convex Function Definition:

$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y), \forall \alpha + \beta = 1, \alpha, \beta \ge 0$

B. Convex Set Definition: $\forall x, y \in C$

We have $\alpha x + \beta y \in C, \forall \alpha + \beta = 1, \alpha, \beta \ge 0$



Chapter 2 Convex Set

1. Set Convexity and Specification
   i. Convexity
   ii. Implicit vs. Explicit Enumeration
2. Convex Set Terms and Definitions
3. Separating Hyperplanes
4. Dual Cones

1. Set Convexity and Specification: Convexity

A set S is convex if we have

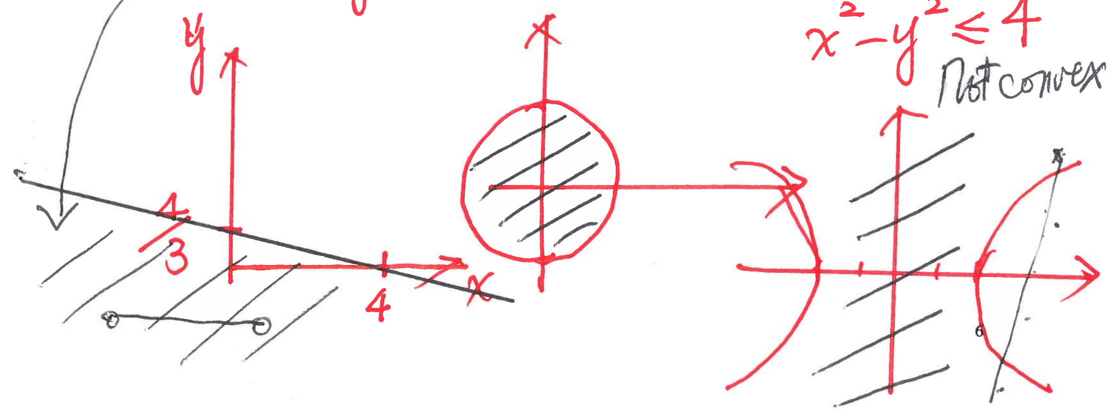
$\alpha x + \beta y \in S, \forall \alpha + \beta = 1, \alpha, \beta \geq 0, \forall x, y \in S$

Examples:

Convex

$x+3y \leq 4$ $\quad x^2+y^2 \leq 4$ $\quad$ Not convex $x^2+y^2 > 4$

$x^2-y^2 \leq 4$ Not convex



1. Set Convexity and Specification: Convexity

A set S is convex if we have

$$\alpha x + \beta y \in S, \forall \alpha + \beta = 1, \alpha, \beta \geq 0, \forall x, y \in S$$

Remark: The definition implies the following

1. Most used sets in the class
   1. Scalar set: $S \subset R$ $\{1 \leq x \leq 4\}$
   2. Vector set: $S \subset R^n$ $\quad x_1 + 2x_2 + 3x_3 \leq 1$
   3. Matrix set: $S \subset R^{n\times m}$ $\quad \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$
2. Set S is convex if every two points in S has the connected straight segment in the set.
3. For convex sets S_1 and S_2: $S_1 \cap S_2$ is also convex

$\alpha M_1 + \beta M_2 \in S$

$\forall M_1, M_2 \in S$

$\alpha + \beta = 1$

$\alpha, \beta \geq 0.$

If $x \in S_1 \cap S_2 \Rightarrow x \in S_1, x \in S_2$

$\quad y \in S_1 \cap S_2 \Rightarrow y \in S_1, y \in S_2$ $\quad \Rightarrow \alpha x + \beta y \in S_1$

$\alpha x + \beta y \in S_2$

Then $\alpha x + \beta y \in S_1 \cap S_2$ $\forall \alpha + \beta = 1, \alpha, \beta \geq 0$

1. Set Convexity and Specification:

Set Specification via Implicit or Explicit Enumeration

Element $\quad$ quantification

Implicit Expression	$S_I = \{x \mid Ax \leq b, x \in R^n\}$
Explicit Enumeration	$S_E = \{Ax \mid x \in R^n_+\}$

Implicit Expression:
Constraints
Min $f_o(x)$
Subject to
$\quad Ax \leq b, x \in R^n$

Explicit Expression:
Enumeration
Min $f_o(Ax), x \in R^n_+$

Implicit Expression $S_1 = \{x|Ax \le b\}$

Example: $\{x|Ax\le b\}$

$$
\begin{aligned}
x_1 \quad +2x_2 \quad +3x_3 &\le 4\\
2x_1 \quad -x_2 \qquad\qquad &\le 3\\
x_2 \quad +x_3 &\le 5\\
x_3 &\le 10
\end{aligned}
$$

Remark: Simultaneous linear constraints imply AND, Intersection of the constraints

Simultaneous linear (equations) inequalities form a convex set

$$A = \begin{bmatrix} 1 & 2 & 3\\ 2 & -1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix}, b = \begin{bmatrix} 4\\ 3\\ 5\\ 10 \end{bmatrix}$$

$S_1 = \{x|Ax \le b, x \in R^n\}$ is a convex set

Proof: Given two vectors $u, v \in S_1$, i.e. $Au \le b$, $Av \le b$ $\Rightarrow \forall u, v \in S_1$

For $w = \theta_1 u + \theta_2 v, \forall \theta_1 + \theta_2 = 1, \theta_1, \theta_2 \ge 0$

we have $Aw \le b$.

$(Aw = \theta_1 Au + \theta_2 Av \le \theta_1 b + \theta_2 b = b)$ $\Rightarrow w = \theta_1 u + \theta_2 v \in S_1$

The inequality implies $w \in S_1$

By definition, set S_1 is convex.

Remark:

1. Simultaneous linear constraints imply AND, Intersection of the constraints
2. Linear constraints constitute a convex set.

Example: $S_1 = \{x|Ax\le b, x\in R^n\}$

$S_2 = \{x|Ax \ge b, x \in R^n\}$ convex? yes

$S_3 = \{x|Ax = b, x \in R^n\}$ convex? yes

If $Ax \ge b$, then we can write $-Ax \le b$

Let $A' = -A$, then we have $A'x \le -b$.

If $Ax = b$, $\Rightarrow Ax \le b$, & $-Ax \le -b$, Let $A' = \begin{bmatrix} A\\ -A \end{bmatrix}$, $A'x \le \begin{bmatrix} b\\ -b \end{bmatrix}$

freq. domain time domain

Example: $S = \{x \in R^m | |p_x(t)| \le 1 \text{ for } |t| \le \frac{\pi}{3}\}$ ~~is convex~~

where $p_x(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

If $x, y \in S$ $(t, x_i \in R)$

i.e. $p_x(t) = x_1 \cos t + \cdots + x_m \cos mt \le 1, \forall t$.

$p_y(t) = y_1 \cos t + \cdots + y_m \cos mt \le 1, t$

$p_{\theta x + (1-\theta) y}(t) = (\theta x_1 + (1-\theta) y_1)\cos t + \cdots + (\theta x_m + (1-\theta) y_m)\cos mt$

$= \theta p_x(t) + \theta p_y(t) \le 1$

1. Specification of Set: Explicit Expression

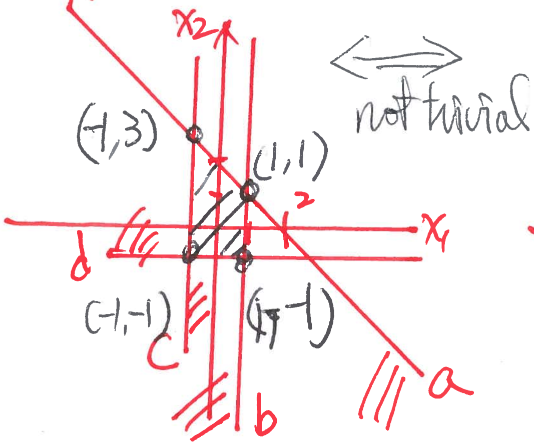
<u>Implicit and Explicit Conversion</u>

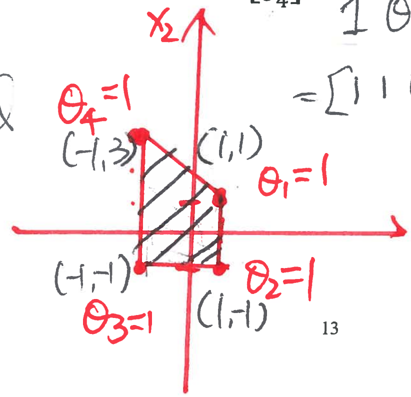
Example: $\{x|Ax \le b, x \in R^n\}$ vs $\{U\theta|\ 1^T\theta = 1,\ \theta \in R_+^m\}$

$$\begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

$$1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad 1^T\theta = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

$\Longleftrightarrow$ not trivial





1. Implicit vs Explicit Enumeration of Convex Set

Remark:
Implicit Expression: Constraints of the problem formulation
Explicit Enumeration: Formulation of the objective function
The interchange may not be trivial.

$$\begin{array}{c} \min f_0(x) \\ s.t. Ax \le b \\ x \in R^n \end{array} \qquad \begin{array}{c} \min f_0(U\theta) \\ s.t.\ I^T\theta \le 1 \\ U \in R^{nm}, \theta \in R_+^m \end{array}$$

Every vector u_i in matrix U is a solution of n equations in constraint $Ax \le b$

~~n~~ p equations, n variables $\Rightarrow$ *comb(~~n~~ p, n) possible vertex points.*

exponential

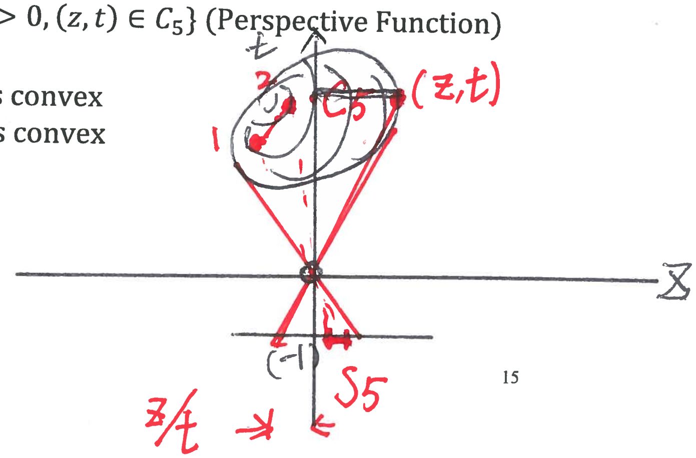
1. Implicit vs Explicit Enumeration of Convex Set

<u>Explicit Enumeration</u> Image process

$$S_4 = \left\{ \frac{Ax+b}{c^Tx+d} \middle| (c^Tx+d) > 0, x \in C_4 \right\} \text{ (Projective Function)}$$

$$S_5 = \left\{ \frac{z}{t} \middle| z \in R^n, t > 0, (z,t) \in C_5 \right\} \text{ (Perspective Function)}$$

S_4 is convex if C_4 is convex
S_5 is convex if C_5 is convex



1. Implicit vs Explicit Enumeration of Convex Set

Statement: S_5 *is convex* if C_5 *is convex.*

Proof: Given $\binom{z_1}{t_1} \in S_5$, $\binom{z_2}{t_2} \in S_5$, let us set

$$z_3 = \alpha z_1 + \beta z_2, t_3 = \alpha t_1 + \beta t_2, \forall \alpha + \beta = 1, \alpha, \beta \ge 0$$

We have $\dfrac{z_3}{t_3} = \dfrac{\alpha z_1 + \beta z_2}{\alpha t_1 + \beta t_2} = \underbrace{\dfrac{\alpha t_1}{\alpha t_1 + \beta t_2}}_{\alpha'}\dfrac{z_1}{t_1} + \underbrace{\dfrac{\beta t_2}{\alpha t_1 + \beta t_2}}_{\beta'}\dfrac{z_2}{t_2}$

Let $\alpha' = \dfrac{\alpha t_1}{\alpha t_1 + \beta t_2}, \beta' = \dfrac{\beta t_2}{\alpha t_1 + \beta t_2}$
(Note that $\alpha' + \beta' = 1, \alpha', \beta' \ge 0$),

we have $\dfrac{z_3}{t_3} = \alpha'\dfrac{z_1}{t_1} + \beta'\dfrac{z_2}{t_2} \in S_5$ $\quad \alpha'+\beta'=1.\ \alpha',\beta' \ge 0$

Therefore, by definition S_5 is convex.