Infeasible Start Newton's Method

$\min f(x)$

$Ax=b \rightarrow r = Ax-b$

The search of the feasible start point,

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ v \end{bmatrix} = -\begin{bmatrix} \nabla f(x) \\ Ax-b \end{bmatrix}$$

$A\Delta x = -Ax+b$

We can write in incremental derivation,

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + A^T v \\ Ax-b \end{bmatrix}$$

r_{dual}

r_{pri}

$A(x+\Delta x)=b$

$$\begin{bmatrix} \nabla^2 f(x) & 0 \\ 0 & -A\nabla^2 f(x)^{-1}A^T \end{bmatrix}$$

KLU, SuperLU

direct matrix solvers.

Newton Method: Infeasible Start

Algorithm.

Given $x \in D, v$, tolerance $\epsilon > 0, \alpha \in \left(0, \frac{1}{2}\right)$, $\beta \in (0,1)$.

$0, \frac{1}{2}$

Repeat

1. Compute primal and dual Newton steps $\Delta x_{nt}, \Delta v_{nt}$
2. Line search on $||r(x,v)||_2 = ||(r_{dual}(x,v), r_{pri}(x,v)||_2$

   $t := 1$

   while $||r(x+t\Delta x_{nt}, v + t\,\Delta v_{nt})||_2 > (1-\alpha t)||r(x,v)||_2$

   $\quad t := \beta t.$
3. Update $x := x + t\Delta x_{nt}, v := v + t\,\Delta v_{nt}$

Until $Ax = b$ and $||r(x,v)||_2 \le \epsilon$

Summary

KKT Linear Equations:
Quadratic objective function + linear equality constraints

Newton's Method:
Twice differentiable obj function + linear equality constraints

Interior Point Method:
Twice differentiable obj function + linear equality + inequality constraints

CSE203B Convex Optimization
Chapter 11 Interior Point Methods

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Chapter 11: Interior-Point Methods

- Formulation
  - Inequality constrained optimization
- Barrier Method
- Generalized Inequalities Problems
- Primal Dual Interior Point Methods
- Summary

Formulation: The problem

Problem: $\min f_0(x)$

Subject to $f_i \leq 0, \ i = 1, \dots, m$

$Ax = b$

Function f_is are convex, twice continuously differentiable

We assume that rank $A = p, A \in R^{p \times n}$.

Issues:

1. m can be exponential.
2. When to put $f_i = 0$ (active)? There are 2^m combination.

Formulation: logarithmic barrier

Problem:

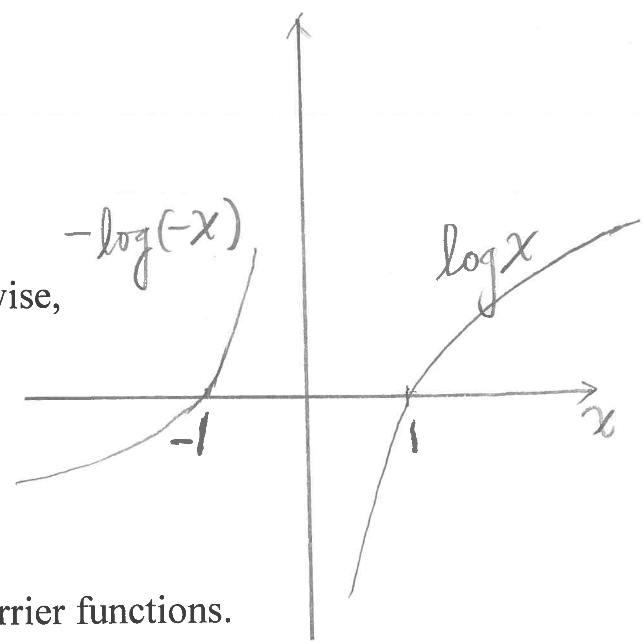
$\min f_0(x) + \sum_{i=1}^{m} I_{f_i(x)}$

$s.t. \quad Ax = b$

When $I_u = 0$ if $u \leq 0$, $I_u = \infty$. Otherwise,

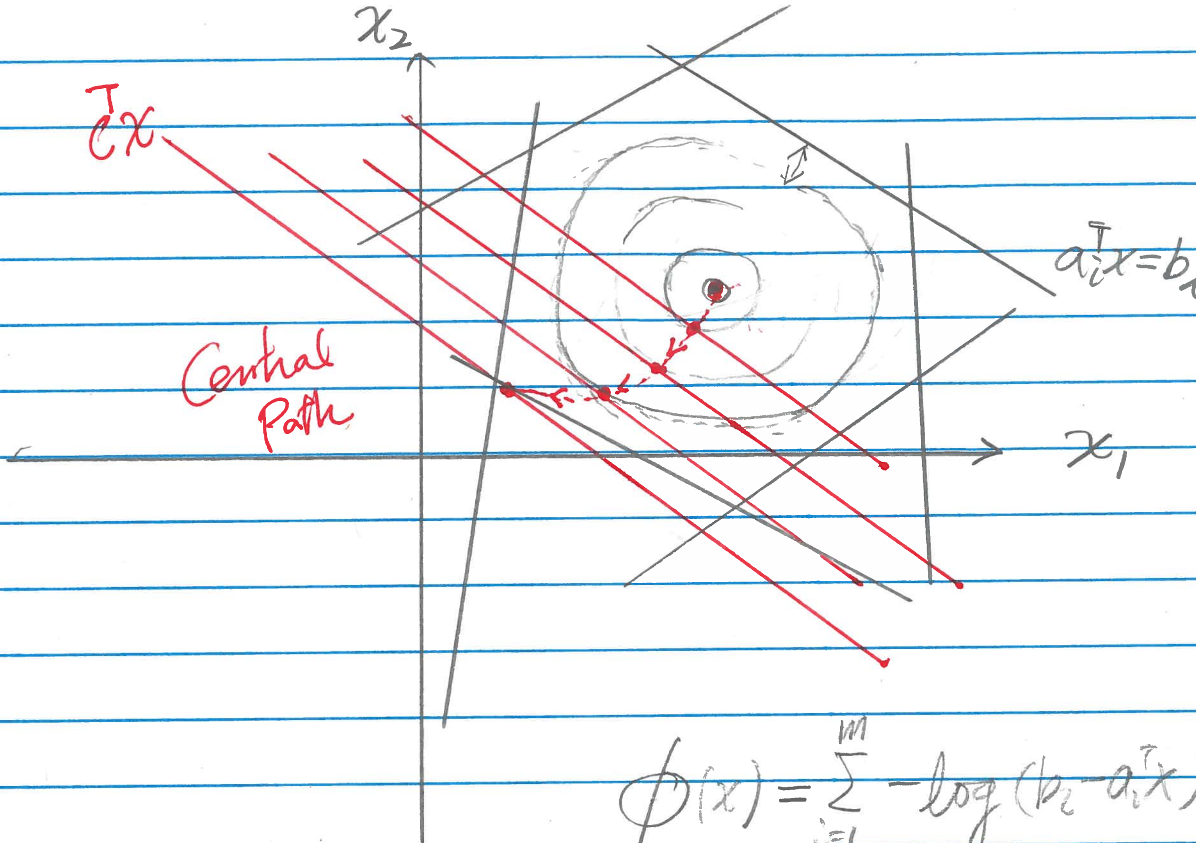
$min\, f_0(x) + \frac{-1}{t} \sum_{i=1}^{m} \log(-f_i(x))$

$s.t. \quad Ax = b$



Remark:

1. Convert inequality constraints to barrier functions.
2. Incorporate barrier functions in objective function.
3. Increase t to improve accuracy.



$$\phi(x) = \sum_{i=1}^{m} -\log(b_i - a_i^T x)$$

$$\nabla\phi(x) = \sum_{i=1}^{m} -\frac{a_i}{b_i - a_i^T x}$$

Formulation: logarithmic barrier

Let us set

$$\phi(x) = -\sum_{i=1}^{m} log(-f_i(x)), \qquad dom\ \phi = \{x | f_i(x) < 0\}$$

$\phi(x)$ *is convex and twice differentiable*

$$\nabla\phi(x) = \sum_{i=1}^{m} -\frac{1}{f_i(x)} \nabla f_i(x)$$

$$\nabla^2\phi(x) = \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T - \frac{1}{f_i(x)} \nabla^2 f_i(x)$$

Central Path is $\{x^*(t) | t > 0\}$

$$\begin{aligned} &min \quad t f_0(x) + \phi(x) \\ &s.t. \quad Ax = b \end{aligned}$$

Formulation: logarithmic barrier

Ex:

Problem: $\quad min \quad c^T x$

$\qquad s.t. \quad a_i^T x \leq b_i, \qquad i = i, \ldots, m$

Log barrier formulation:

$$\min t c^T x - \sum_{i=1}^{m} \log(b_i - a_i^T x)$$

Hyperplane $c^T x = c^T x^*(t)$ *is tangent to real curve* φ *through* $x^*(t)$.

Solution $x^*(t)$ *balance the force between* $-t\nabla f_0(x)$ *and* $\sum_{i=1}^{m} -\frac{1}{f_i(x)} \nabla f_i(x)$.

$$tc + \sum_{i=1}^{m} -\frac{1}{(b_i - a_i^T x)} a_i = 0$$

Formulation: logarithmic barrier

Ex:

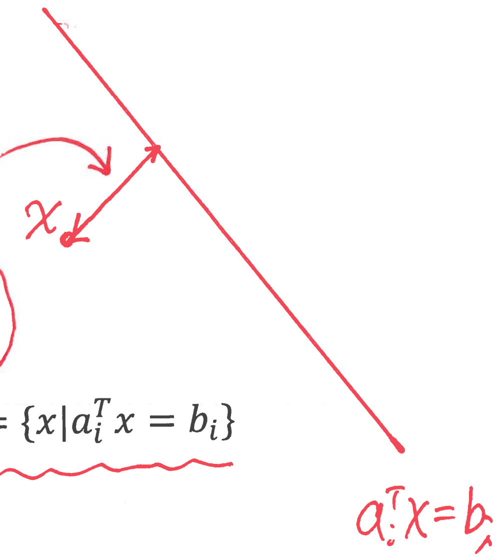
Problem: $min \quad c^T x$

$s.t. \quad a_i^T x \le b_i \qquad i = i, \dots, m$

$-t\,\nabla f_0(x) = -tc$

$$\sum_{i=1}^{m} -\frac{1}{f_i(x)} \nabla f_i(x) = \sum_{i=1}^{m} -\frac{1}{b_i - a_i^T x} a_i$$

Note that min $\left\| \frac{1}{b_i - a_i^T x} a_i \right\|_2 = \frac{1}{dist(x_i H_i)}$, $H_i = \{x | a_i^T x = b_i\}$



Barrier Method: Algorithm

Given strictly feasible x, $t = t^0 > 0, \mu > 1, \epsilon > 0$

Repeat (10-20)

1. Centering step to find solution $x^*(t)$

   Problem: $min \quad tf_0(x) + \phi(x)$ *(Newton's method)*

   $s.t. \quad Ax = b$

2. Update $x = x^*(t)$

3. Stopping criterion: exit if $\frac{m}{t} < \epsilon$

4. Increase $t = \mu t$

Complexity: # Repeats (Outer iterations) = $\frac{\log(\frac{m}{\epsilon t^{(0)}})}{log\mu}$

Plus the initial centering step $x^*(t^{(0)})$

Barrier Method: Newton's Step for Modified KKT

$$\begin{bmatrix} t\nabla^2 f_0(x) + \nabla^2\phi(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ v \end{bmatrix} = -\begin{bmatrix} t\nabla f_0(x) + \nabla\phi(x) \\ 0 \end{bmatrix}$$

$$\nabla \sum_{i=1}^{m} (-\log(-f_i(x))) = \sum_{i=1}^{m} -\frac{1}{f_i(x)} \nabla f_i(x)$$

$$\nabla^2 \sum_{i=1}^{m} (-\log(-f_i(x)))$$

$$= \sum_{i=1}^{m} [-\frac{1}{f_i(x)} \nabla^2 f_i(x) + \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T]$$

Barrier Method: Central Path

Min $f_0(x) + \frac{-1}{t}\sum_{i=1}^{m} \log(-f_i(x))$

$s.t. \quad Ax = b$

Lagrangian: $L(x, v) = f_0(x) + \frac{-1}{t}\sum_{i=1}^{m} \log(-f_i(x)) + v^T(Ax - b)$

For an optimal solution, we have $(x^*(t), \bar{v}(t))$

$\nabla f_0(x^*) + \sum -1/(tf_i(x^*))\ \nabla f_i(x^*) + A^T\bar{v} = 0$

We can view the dual points from central path:

$$\lambda_i^*(t) = -1/(tf_i(x^*)), i = 1, \dots, m$$

Original Lagrangian:

$$L(x, \lambda, v) = f_0(x) + \sum \lambda_i f_i(x) + v^T(Ax - b)$$

Replace with $(x^*(t), \lambda^*(t), \bar{v}(t))$:

$L(x^*, \lambda^*, \bar{v}) = f_0(x^*) + \sum \lambda_i^* f_i(x^*) + \bar{v}^T(Ax^* - b) = f_0(x^*) - \dfrac{m}{t}$

Thus, we have $f_0(x^*(t)) - p^* \leq m/t$

$p^* = \max_{\lambda, v} g(\lambda, v) \geq g(\lambda^*, \bar{v}) = \min_{x} L(x, \lambda^*, \bar{v}) = f_0(x^*) - \frac{m}{t}$

Barrier Method: Feasible Solution Search

Search 1:
min s
$s.t. f_i(x) \le s, i = 1, \dots, m$
$Ax = b, s \in R$

Search 2:
min $1^T s$, $s \in R_+^m$
$s.t. f_i(x) \le s_i, i = 1, \dots, m$
$Ax = b$

Barrier Method: complexity analysis

#Repeats (outer iterations)
=Ceiling(log($m/(\epsilon t^0)$)/$log\mu$)
#Newton steps per outer iteration (self-concordance)
$= \frac{m(\mu - 1 - log\mu)}{\gamma} + \log_2 \log_2 1/\epsilon_{nt}$,
where $\gamma = \alpha\beta(1-2\alpha)^2/(20-8\alpha)$