CSE203B Convex Optimization:
Chapter 9: Unconstrained Minimization

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Chapter 9 Unconstrained Minimization

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Introduction

Problem: min $f(x)$ where $f: R^n \to R$ is convex and twice continuously differentiable

Theorem: Necessary and sufficient condition for a point x^* to be optimal is $\nabla f(x^*) = 0$.

Remark: keywords Taylor's expansion

Taylor's Expansion & Bounds: Scalar case

$f(x) = f(x_0) + \nabla f(x_0)^T(x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(z)(x - x_0)$

for some z on the segment $[x, x_0]$

(1)Scalar case: $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(z)(x - x_0)^2$

We simplify the notations $f(x) = \frac{m}{2}(x - x_0)^2 + a(x - x_0) + b$

For fixed $m, a, and\ b$, the optimal solution can be derived as:

$$\nabla f(x) = 0 \Rightarrow m(x - x_0) + a = 0 \Rightarrow x - x_0 = -\frac{a}{m}$$

Thus, we have

$$f(x) = \frac{m}{2}\frac{a^2}{m^2} + a\frac{(-a)}{m} + b = \frac{a^2}{2m} - \frac{a^2}{m} + b = \frac{-a^2}{2m} + b$$

Or $f(x) - f(x_0) = -\frac{a^2}{2m}$

a. How far from opt. x^**?* $x^* - x_0 = -\frac{a}{m}$

b. How much difference from opt. $f(x^*)$*?* $f(x_0) - f(x^*) = \frac{a^2}{2m}$

Taylor's Expansion & Bounds: Example

$$f(x) = x^2 + 4x + 1$$

For the format

$$f(x) = \frac{m}{2}x^2 + ax + b, \qquad m = 2\,, a = 4, b = 1.$$

Let $x_0 = 0$, we have the answer.

a. *How far?* $x^* - x_0 = -\frac{a}{m} = -2$

b. *How much?* $f(x_0) - f(x^*) = \frac{a^2}{2m} = 4$

Taylor's Expansion & Bounds: Bounds

(2) Vector case:

Assumption A: $\nabla^2 f(x)$ is bounded, *i.e.* $mI \preccurlyeq \nabla^2 f(x) \preccurlyeq MI$

Theorem A: We have the following bounds

$$\frac{1}{M}\|\nabla f(x_0)\|_2 \overset{④}{\leq} \|x_0 - x^*\|_2 \overset{①}{\leq} \frac{2}{m}\|\nabla f(x_0)\|_2$$

$$\frac{1}{2M}\|\nabla f(x_0)\|_2^2 \overset{③}{\leq} f(x_0) - p^* \overset{②}{\leq} \frac{1}{2m}\|\nabla f(x_0)\|_2^2$$

Proof:①

$$f(x) + \nabla f(x)^T(y-x) + \frac{m}{2}\|y-x\|_2^2 \leq f(y)$$

$$\leq f(x) + \nabla f(x)^T(y-x) + \frac{M}{2}\|y-x\|_2^2$$

(Taylor's Expansion + Assumption A)

Taylor's Expansion & Bounds: Bounds

***Proof* ①:** $\|x - x^*\|_2 \le \frac{2}{m}\|\nabla f(x)\|_2$

$p^* = f(x^*) \ge f(x) + \nabla f(x)^T(x^* - x) + \frac{m}{2}\|x^* - x\|_2^2$ *(Taylor's exp + Assumption A.)*

$$\ge f(x) - \|\nabla f(x)\|_2\|x^* - x\|_2 + \frac{m}{2}\|x^* - x\|_2^2$$

We shift f(x) to the left hand side.

$0 \ge p^* - f(x) \ge -\|\nabla f(x)\|_2\|x^* - x\|_2 + \frac{m}{2}\|x^* - x\|_2^2$

Shift $-\|\nabla f(x)\|_2\|x^* - x\|_2$ *to the left,*

$\|\nabla f(x)\|_2\|x^* - x\|_2 \ge \frac{m}{2}\|x^* - x\|_2^2$

Therefore we have

1. $\|\nabla f(x)\|_2 \ge \frac{m}{2}\|x^* - x\|_2$

2. $\|x^* - x\|_2 \le \frac{2}{m}\|\nabla f(x)\|_2$

Taylor's Expansion & Bounds: Bounds

Proof ②: $f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{m}{2}\|y - x\|_2^2$

(Taylor's exp + assumption A.)

$\geq f(x) - \frac{1}{2m}\|\nabla f(x)\|_2^2$ *(Minimization with y)*

Thus, we have

$$f(x) - f(y) \leq \frac{1}{2m}\|\nabla f(x)\|_2^2, \qquad \forall y$$

Therefore

$$f(x) - f(x^*) \leq \frac{1}{2m}\|\nabla f(x)\|_2^2$$

Taylor's Expansion & Bounds: Bounds

Proof ③**:** $f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} \|y - x\|_2^2$

(Taylor's exp + assumption A.)

$\geq f(x) - \frac{1}{2m} \|\nabla f(x)\|_2^2$ *(Minimization with y)*

Let $y = x - \frac{1}{M} \nabla f(x)$, *we have*

$$f\left(x - \frac{1}{M} \nabla f(x)\right) \leq f(x) + \nabla f(x)^T \frac{-1}{M} \nabla f(x) + \frac{M}{2} \left\| \frac{1}{M} \nabla f(x) \right\|_2^2$$

$$= f(x) - \frac{1}{2M} \|\nabla f(x)\|_2^2$$

Shift the terms on the left and right, we have

$$\frac{1}{2M} \|\nabla f(x)\|_2^2 \leq f(x) - f\left(x - \frac{1}{M} \nabla f(x)\right)$$

$$\leq f(x) - f(x^*)$$

Taylor's Expansion & Bounds: Bounds

(4) Proof: $f(y) \le f(x) + \nabla f(x)^T(y-x) + \frac{M}{2}\|y-x\|_2^2$

(Taylor's exp. + assumption A)

(i) Let $x = x^*$, we have $\nabla f(x^*) = 0$,

thus, we can write the above eq.

$$f(y) \le f(x^*) + \frac{M}{2}\|y-x^*\|_2^2$$

or $$f(y) - p^* \le \frac{M}{2}\|y-x^*\|_2^2$$

(ii) From (3), we have

$$\frac{1}{2M}\|\nabla f(x_o)\|_2^2 \le f(x_o) - p^*$$

(iii) From (i)&(ii), we have

$$\frac{1}{2M}\|\nabla f(x_o)\|_2^2 \le \frac{M}{2}\|x_o - x^*\|_2^2$$

Therefore, we have

$$\frac{1}{M}\|\nabla f(x_o)\|_2 \le \|x_o - x^*\|_2$$

Taylor's Expansion & Bounds

Remark:

(1) If $\|\nabla f(x)\|_2 \leq (2m\epsilon)^{\frac{1}{2}}$

We have $\|x - x^*\|_2 \leq \frac{2}{m}(2m\epsilon)^{\frac{1}{2}}$

$$f(x) - f(x^*) \leq \frac{\|\nabla f(x)\|_2^2}{2m} = \epsilon$$

(2) The bounds can be used to design algorithms.

prove the convergence.

(3) If $M \gg m$ $(e.g. 10^{10})$

Impact on the bounds become very loose

→ Efficiency of gradient descent approaches.

(4) Quadratic obj. with sparse matrix (A)

$\frac{1}{2}x^T Ax + b^T x + c$

is a preferred formulation in terms of algorithm efficiency.

II. Descent Methods

Given convex function, twice continuously differentiable $f(x)$ and an initial point $x_o \in dom\, f$.

Repeat

1. Determine a descent direction Δx ($\nabla f(x)^T \Delta x < 0$)
2. Line Search, choose a step size $t > 0$.
3. Update $x = x + t\Delta x$

Until stopping criterion is met.

Line Search : $t = arg \min_{t>0} f(x + t\Delta x)$

Backtracking line search ($\alpha \in (0, 1/2), \beta \in (0, 1)$)

Start at $t = 1$, repeat $t := \beta t$

until $f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$

Stopping criterion $\|\nabla f(x)\|_2 \leq \eta \quad \eta = (2m\epsilon)^{\frac{1}{2}}$ (Theorem A (2))

II. Descent Methods: Example

Problem: min $f(x) = \frac{1}{2}(x_1^2 + \gamma x_2^2)$ $\gamma > 0$

$$x^o = (\gamma, 1), f(x^0) = \frac{\gamma(\gamma+1)}{2}, \nabla f(x^0) = (\gamma, \gamma)$$

Thus, $x^1 = (\gamma, 1) - t(\gamma, \gamma) = (\gamma(1-t), 1 - t\gamma)$

and $\nabla f(x^1) = (\gamma(1-t), \gamma(1-t\gamma))$

1. To opt $f(x^1)$ with respect to variable t,

we have $f(x^1) = \frac{1}{2}(\gamma^2(1-t)^2 + \gamma(1-t\gamma)^2)$

$$\frac{\partial f(x^1)}{\partial t} = \gamma^2(1-t) + \gamma(1-t\gamma)\gamma = 0$$

Thus, $t = \frac{2\gamma^2}{\gamma^2+\gamma^3} = \frac{2}{1+\gamma}$, and $x^1 = \left(\frac{\gamma(\gamma-1)}{1+\gamma}, \frac{1-\gamma}{1+\gamma}\right) = \left(\frac{10\times 9}{11}, -\frac{9}{11}\right)$

2. We repeat the process to step k, $x^k = (\gamma\left(\frac{\gamma-1}{\gamma+1}\right)^k, \left(\frac{1-\gamma}{1+\gamma}\right)^k)$

3. Equal potential plot

$$f(x^k) = \frac{\gamma(\gamma+1)}{2}\left(\frac{\gamma-1}{\gamma+1}\right)^{2k} = \left(\frac{\gamma-1}{\gamma+1}\right)^{2k} f(x^o) = \left(\frac{1-m/M}{1+m/M}\right)^{2k} f(x^o)$$

II. Descent Methods: Descent for various norms

1. Problem: Min $f(x)$
2. For each iteration, we try the steepest descent in terms of a given norm.

$$\text{Min } \nabla f(x)^T \Delta x$$
$$\text{s.t. } \| \Delta x\| \leq 1$$

3. We show the step of
i. Quadratic norm
ii. L1 norm

II. Descent Methods: Descent for quadratic norm

1. Problem: Min $f(x)$
2. For each iteration, we try the steepest descent in terms of a given norm.

$$Min_{\Delta x} \ \nabla f(x)^T \Delta x$$

$$\text{s.t.} \parallel \Delta x \parallel_P \leq 1$$

$||\Delta x||_P = (\Delta x^T P \Delta x \ )^{1/2}$, $P \in S_{++}^n$

Lagrangian $L(\Delta x, \lambda) = \nabla f(x)^T \Delta x + \lambda \ (\parallel \Delta x \parallel_P - 1)$, $\lambda \geq 0$

We can derive: $\Delta x_{nsd} = -\left(\nabla f(x)^T P^{-1} \nabla f(x)\right)^{-1/2} P^{-1} \nabla f(x)$

Or $\Delta x_{sd} = -P^{-1} \nabla f(x)$

II. Descent Methods: Descent for quadratic norm

The coordinate change has effects on the descent direction.

Example: min $f(x) = \frac{1}{2}x^T Px + q^T x, P \in S_{++}^n$

Affine transform: $\bar{x} = P^{1/2}x$

II. Descent Methods: Descent for L1 norm

1. Problem: Min $f(x)$
2. For each iteration, we try the steepest descent in terms of a given norm.

$$\text{Min } \nabla f(x)^T \Delta x < 0$$
$$\text{s.t. } \| \Delta x \|_1 \leq 1$$

Lagrangian $L(\Delta x, \lambda) = \nabla f(x)^T \Delta x + \lambda \left(\| \Delta x \|_1 - 1\right), \lambda \geq 0$

We can derive: $\Delta x_{nsd} = -sign\left(\frac{\partial f(x)}{\partial x_i}\right) e_i$,

where i is the index for which $\left\| \nabla f(x) \right\|_\infty = |\nabla f(x)_i|$

Or $\Delta x_{sd} = -\frac{\partial f(x)}{\partial x_i} e_i$

Gradient descent method: Convergence analysis

$$\tilde{f}(t) \equiv f\left(x - t\nabla f(x)\right) \le f(x) - t\|\nabla f(x)\|_2^2 + \frac{Mt^2}{2}\|\nabla f(x)\|_2^2$$

$$\tilde{f}(t_{exact}) \le \tilde{f}\left(t = \frac{1}{M}\right) \le f(x) - \frac{1}{2M}\|\nabla f(x)\|_2^2 \quad (\min_t f(x)\nabla f(x))$$

A. $\tilde{f}(t_{exact}) - p^* \le f(x) - p^* - \frac{1}{2M}\|\nabla f(x)\|_2^2$

B. $\frac{1}{2M}\|\nabla f(x)\|_2^2 \ge \frac{m}{M}(f(x) - p^*)$ since $\frac{\|\nabla f(x)\|_2^2}{2m} \ge f(x) - p^*$

C. From B, we have

$$f(x) - p^* - \frac{1}{2M}\|\nabla f(x)\|_2^2 \le f(x) - p^* - \frac{m}{M}(f(x) - p^*)$$
$$= (f(x) - p^*)(1 - \frac{m}{M})$$

D. We can conclude from A & C

$$f\left(x^{k+1}\right) - p^* \le \left(1 - \frac{m}{M}\right)\left(f\left(x^k\right) - p^*\right) \le \left(1 - \frac{m}{M}\right)^k (f(x^o) - p^*)$$

To achieve $f(x^*) - p^* \le \epsilon$,

we need $\frac{\log\left((f(x^o) - p^*)/\epsilon\right)}{\log(1/c)}$ $steps$, where $c = 1 - \frac{m}{M} < 1$,

Gradient descent method : Convergence analysis

$\log(1/c) = -\log(1 - m/M) \approx m/M$ for large M/m

Remark: when $M/m > 100$

the method can be very slow.

Newton Step

Use the approximation of 2nd order Taylor′s Exp.

$$f(x+v) \approx f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v$$

We would like to derive

$$\nabla_v f(x+v) = 0 \ \rightarrow \nabla f(x) + \nabla^2 f(x) v = 0$$

Thus, we have $v = -\nabla^2 f(x)^{-1} \nabla f(x)$

$$f(x+v) = f(x) + (-1)\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x) + \frac{1}{2} \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$$

$$= f(x) - \frac{1}{2} \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$$

Input $x \in dom\, f,\ \epsilon > 0$

Repeat: 1. $\Delta x_{nt} \coloneqq -\nabla^2 f(x)^{-1} \nabla f(x), \lambda^2(x) = \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$

2. $Quit\ if\ \lambda^2/2 \leq \epsilon$

3. $Line\ Search\ t$

4. $x \coloneqq x + t\Delta x_{nt}$

Newton Method : Convergence analysis

Assumptions: $S = \{x \in dom\ f | f(x) \le f(x_o)\}$

f strongly convex on S with constant m, s.t. $\nabla^2 f(x) \ge mI, \forall x \in S$

$\nabla^2 f$ is Lipschitz continuous on S with constant L, i.e.

$$\|\nabla^2 f(x) - \nabla^2 f(y)\|_2 \le L\|x - y\|_2$$

Outlines: $\exists \eta \in (0, m^2/L)$, two cases.

1. Damped Newton Phase: ($t < 1$)

$\|\nabla f(x)\|_2 \ge \eta$ then $f\left(x^{k+1}\right) - f\left(x^k\right) \le -\alpha\beta\eta^2 m/M^2$

2. Pure Newton Phase (Quadratically Convergent Stage): ($t = 1$)

$\|\nabla f(x)\|_2 < \eta$ then

$$\frac{L}{m^2}\left\|\nabla f\left(x^{k+1}\right)\right\|_2 \le \left(\frac{L}{2m^2}\left\|\nabla f\left(x^k\right)\right\|_2\right)^2$$

$$\le \left(\frac{L}{2m^2}\left\|\nabla f\left(x^l\right)\right\|_2\right)^{2^{k+1-l}} \le \left(\frac{1}{2}\right)^{2^{k+1-l}} \qquad k + 1 \ge l$$

Newton Method: Affine Invariant

Problem: $\min f(x)$

Theorem: Newton's step is invariant to affine transform.

Proof: Let $x = Ty, T \in R^{nn}, f(x) = f(Ty) = \bar{f}(y)$

For the x coordinate system, we have.

$\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$

Therefore, we have the invariant results

$x + \Delta x_{nt} = T(y + \Delta y_{nt})$.

For the y coordinate system, we have.

1. $\nabla_y \bar{f}(y) = T^T \nabla_x f(Ty)$,

   $\nabla_y^2 \bar{f}(y) = T^T \nabla^2 f(Ty) T$

2. The Newton step at y,

Δy_{nt}
$= -\nabla_y^2 \bar{f}(y)^{-1} \nabla_y \bar{f}(y)$
$= -(T^T \nabla^2 f(x) T)^{-1} (T^T \nabla f(x))$
$= -T^{-1} \nabla^2 f(x)^{-1} \nabla f(x)$
$= T^{-1} \Delta x_{nt}$

Summary

1. Gradient Descent Method: (minimization solution)
   1. Vector operations per iteration
   2. Linear convergence rate
2. Newton’s Method: (equality solution)
   1. Matrix operations per iteration
   2. Quadratic convergence rate (near the solution)
3. Gradient Descent Method Variations:
   1. Conjugate gradient method
   2. Nesterov gradient descent method
   3. Quasi-Newton method