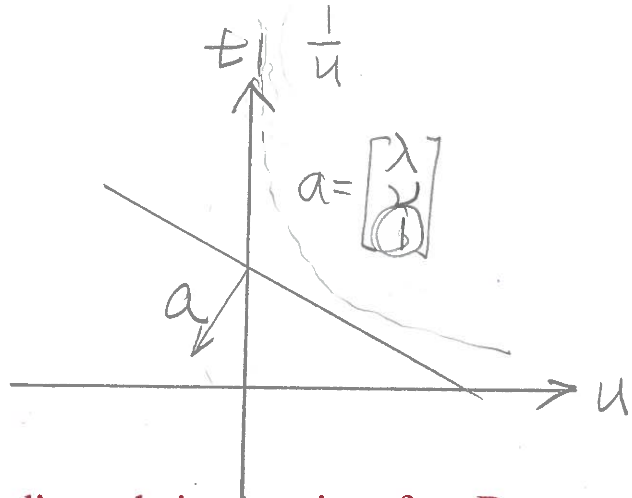
Lagrange dual problem

$$\max g(\lambda, v)$$
$$s.t. \quad \lambda \geq 0$$



Properties
This is a convex problem.
The opt. solution is denoted as d^*
$p^* - d^* = gap \geq 0$
If $gap > 0$, it is a weak duality.
If $gap = 0$, it is a strong duality.

relint: relative interior of set D

Slater's condition
Given that the primal problem is convex,

If $f_i(x) < 0, i = 1, \ldots, m, \exists x \in relint\ D$
Then strong duality holds.

$$relint\ C = \{x \in C | B(x, r) \cap aff\ C \subseteq C\ for\ some\ r > 0\}$$
$$B(x, r) = \{y | \|y - x\| \leq r\}$$

any norm

Shadow Price Interpretation: Food vs. Vitamin

Flour	protein powder
Veg.	vitamins A,B,D,E,K
Fruits	minerals

Primal $\min c^T x \quad s.t.\ Ax \succcurlyeq b,\ x \succcurlyeq 0$ $\qquad$ $\min c^T x \quad s.t.\ -Ax + b \preccurlyeq 0,\ -x \preccurlyeq 0$

$$\min c_1 x_1 + c_2 x_2 + c_3 x_3 \quad \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \succcurlyeq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, x_i \geq 0, \forall i$$

Dual $\max \lambda^T b \quad s.t.\ A^T \lambda \preccurlyeq c,\ \lambda \succcurlyeq 0$

$$\max \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$
$$\text{s.t.} \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \preccurlyeq \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Lagrangian
$$L(x, \lambda) = c^T x + \lambda_I^T(-Ax + b) + \lambda_{II}^T(-x)$$
$$= [c^T + \lambda_I^T(-A) - \lambda_2^T]x + \lambda_I^T b$$
$$c^T = \lambda_I^T(A) + \lambda_{II}^T,\ or\ A^T \lambda_I \leq c$$

Shadow Price Interpretation: Spring Energy & Force

$$min\, f_o(x_1, x_2) = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3(l - x_2)^2$$

f_o : potential energy $\quad k_i > 0$: stiffness constant of spring i

$w/2 - x_1 \leq 0$

$w + x_1 - x_2 \leq 0$

$w/2 - l + x_2 \leq 0$

$$min\, \frac{1}{2}(k_1x_1^2 + k_2(x_2 - x_1)^2 + k_3(l - x_2)^2)$$

$\lambda_1 \quad w/2 - x_1 \leq 0$

$\lambda_2 \quad w + x_1 - x_2 \leq 0$

$\lambda_3 \quad w/2 - l + x_2 \leq 0$

$\lambda_1(w/2 - x_1) = 0, \lambda_2(w - x_2 + x_1) = 0, \lambda_3(w/2 - l + x_2) = 0$

zero gradient condition

$$\begin{bmatrix} k_1x_1 - k_2(x_2 - x_1) \\ k_2(x_2 - x_1) - k_3(l - x_2) \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

λ_i : contact forces between the walls & blocks

KKT (Karush-Kuhn-Tucker) Conditions

2. $f_i(x), i = 1, \dots, m, \;\; h_i(x), \;\; i = 1, \dots, p$ *are differentiable*

1. Primal constraints : $f_i(x) \leq 0, i = 1, \dots, m.$

   $h_i(x) = 0, i = 1, \dots, p.$

2. Dual constraints : $\lambda \succcurlyeq 0$

3. Complementary slackness : $\lambda_i f_i(x) = 0, i = 1, \dots, m.$ If $f_i(x) < 0$, then $\lambda_i = 0$

   Else $f_i(x) = 0$, $\lambda_i > 0$

4. Gradient of Lagrangian with respect to x variables

   $\nabla_x f_o(x) + \sum_{i=1}^{m} \lambda_i \nabla_x f_i(x) + \sum_{i=1}^{p} v_i \nabla_x h_i(x) = 0$

($\nabla_x^2 f_o(x) + \sum_{i=1}^{m} \lambda_i \nabla_x^2 f_i(x) \succcurlyeq 0$ the problem is convex)

Primal Dual Interpretation

Flour, Veg. Fruits $\qquad\qquad$ Protein, VitA, B, E, Minerals

$x_1 \quad x_2 \quad x_3$

$$\min \; c_1x_1 + c_2x_2 + c_3x_3$$

$$\begin{array}{ll} & \text{Flour} \\ \begin{array}{r} \text{Protein } \lambda_1 \\ \text{Vit A } \lambda_2 \\ \text{B } \lambda_3 \\ \text{E } \lambda_4 \\ \text{Minerals } \lambda_5 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \end{array}$$

$$x_1, x_2, x_3 \geq 0$$

$$\max \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4 + \lambda_5 b_5$$

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} & a_{51} \\ a_{12} & a_{22} & a_{32} & a_{42} & a_{52} \\ a_{13} & a_{23} & a_{33} & a_{43} & a_{53} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} \leq \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$$

P.30.

KKT (Karush-Kuhn-Tucker) Conditions

1. Primal, Lagrangian, and Dual

$$\max_{\lambda,v}\min_{x}: (x,\lambda,v) = \max_{\lambda,v} g(\lambda,v)$$

$$\begin{aligned} &\min f_o(x) \\ &f_i \le 0 \\ &h_i = 0 \end{aligned} \qquad L(x,\lambda,v) = f_o(x) + \sum_{i=1}^{m}\lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x), \lambda_i \ge 0$$

$$\min_x \max_{\lambda,v} \qquad \max_{\lambda,v}\min_x \ (dual)$$

1. Feasibility (x,λ,v)
2. $L(x,\lambda,v) = f_o(x) \begin{cases} \lambda_i > 0 \ if \ f_i = 0 \\ \lambda_i = 0 \ if \ f_i < 0 \end{cases}$
3. $g(\lambda,v) = \min_x L(x,\lambda,v) = f_o(x)$

Necessary condition for local optimality
Sufficient when the problem is convex & satisfy regularity conditions (Slater condition)

Sensitivity

Perturbed Problem

$$\begin{aligned} &\min f_o(x) \\ &s.t.\ \ f_i \le u_i \\ &\qquad h_i(x) = w_i \end{aligned}$$

$$\begin{aligned} &\max \tilde{g} = g(\lambda,v) - u^T\lambda - w^T v \\ &s.t.\quad \lambda \ge 0 \end{aligned}$$

$$p^*(u,w) = \max_{\lambda,v} g(\lambda,v) - u^T\lambda - w^T v$$

Unperturbed Problem

$$u_i = w_i = 0$$

$$\begin{aligned} &\max g(\lambda,v) \\ &s.t.\quad \lambda \ge 0 \end{aligned}$$

$$p^*(0,0), \lambda^*, v^*$$

$$p^*(u,w) \ge g(\lambda^*,v^*) - u^T\lambda^* - w^T v^* = p^*(0,0) - \mathrm{u}^T\lambda^* - w^T v^*$$

$$\lambda_i^* = -\frac{\partial p^*(u,w)}{\partial u_i}\bigg| (u,w) = (0,0), \ \ v_i^* = -\frac{\partial p^*(u,w)}{\partial w_i}\bigg| (u,w) = (0,0)$$

Proof : $\dfrac{\partial p^*(0,0)}{\partial u_i} = \lim_{t \searrow 0} \dfrac{p^*(te_i,0) - p^*(0,0)}{t} \ge -\lambda_i^*$

$\dfrac{\partial p^*(0,0)}{\partial u_i} = \lim_{t \nearrow 0} \dfrac{p^*(te_i,0) - p^*(0,0)}{t} \le -\lambda_i^*$

hence, equality

Generalized Inequalities

Primal

$$\min f_o(x)$$
$$f_i \preccurlyeq_{K_i} 0, i = 1, \dots, m$$
$$h_i = 0, i = 1, \dots, p$$

Lagrangian

$$L(x, \lambda, v) = f_o(x) + \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{i=1}^{p} v_i h_i(x),$$
$$\lambda_i \succcurlyeq_{K_i^*} 0, i = 1, \dots, m,\ \lambda_i \in R^{k_i}$$

Lagrange Dual

$$g(\lambda, v) = \inf_x L(x, \lambda, v)$$

Generalized Inequality: KKT Conditions

$$\min f_o(x)$$
$$s.t\ f_i \preccurlyeq_{K_i} 0, i = 1, \dots, m$$
$$h_i = 0, i = 1, \dots, p$$

$f_i(x), i = 1, \dots, m,\ h_i(x),\ i = 1, \dots, p$ are differentiable

1. Primal constraints : $f_i(x) \preccurlyeq_{K_i} 0, i = 1, \dots, m.$
   $h_i(x) = 0, i = 1, \dots, p.$
2. Dual constraints : $\lambda \succcurlyeq_{K_i^*} 0$
3. Complementary slackness : $\lambda_i^T f_i(x) = 0, i = 1, \dots, m.$
4. Gradient of Lagrangian with respect to x variables
   $\nabla_x f_o(x) + \sum_{i=1}^{m} \lambda_i^T \nabla_x f_i(x) + \sum_{i=1}^{p} v_i \nabla_x h_i(x) = 0$

Sensitivity.

$$\min f_0(x)$$

$$f_i(x) \le 0$$

$$h_i(x) = 0.$$

$$\min f_0(x)$$

$$f_i(x) \le u_i$$

$$h_i(x) = w_i.$$

slacks

$$L(x,\lambda,\nu) = f_0(x) + \sum_i \lambda_i f_i + \sum_i \nu_i h_i$$

$$\tilde{L}(\tilde{x},\lambda,\nu) = f_0(x) + \sum_i \lambda_i f_i + \sum \lambda_i h_i - \sum_i \lambda_i u_i - \sum_i \nu_i w_i$$

$$= L(x,\lambda,\nu) - \sum \lambda_i u_i - \sum_i \nu_i w_i$$

$$g(\lambda,\nu) = \min_x L(x,\lambda,\nu)$$

$$\tilde{g}(\lambda,\nu) = \min_x \tilde{L}(\tilde{x},\lambda,\nu)$$

$$= \min_x L(x,\lambda,\nu) - \sum_i \lambda_i u_i - \sum_i \nu_i w_i$$

$$= g(\lambda,\nu) - \sum_i \lambda_i u_i - \sum_i \nu_i w_i$$

$$\max_{\lambda,\nu} g(\lambda,\nu)$$

$$\lambda \ge 0$$

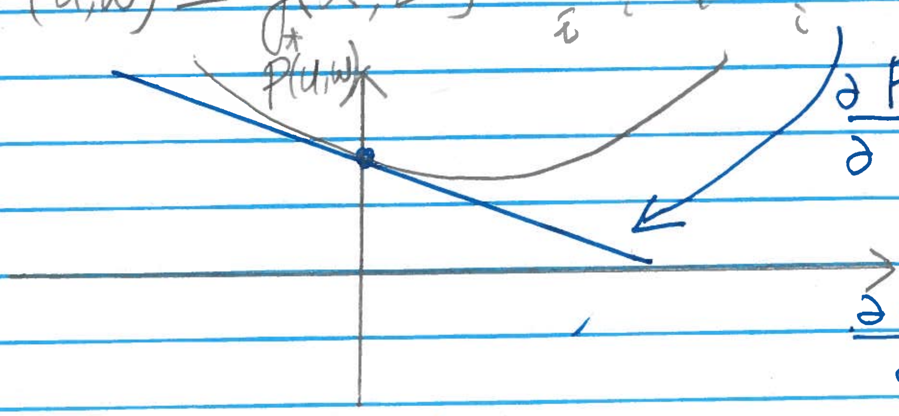
$$p^*(0,0),\ \lambda^*,\ \nu^*$$

$$\max_{\lambda,\nu} \tilde{g}(\lambda,\nu) = g(\lambda,\nu) - \sum \lambda_i u_i - \sum \nu_i w_i$$

$$\lambda \ge 0.$$

$$p^*(u,w)$$

$$p^*(u,w) \ge g(\lambda^*,\nu^*) - \sum_i \lambda_i^* u_i - \sum_i \nu_i^* w_i$$



$$\frac{\partial\, p^*(u,w) - p^*(0,0)}{\partial u_i} \ge -\lambda_i^* \quad \Delta u_i \ge 0$$

$$\frac{\partial\, p^*(u,w) - p^*(0,0)}{\partial u_i} \le -\lambda_i^*$$