Saddle Point: 2-D Table Formulation

Find a saddle point of $f(w, z)$ under constraint I.

Theorem II: A saddle point of 2-D table formulation can be solved if $f(w, z)$ is convex w.r.t. w, and concave w.r.t. z.

Saddle Point:

$\min_{w} w^T A z$

$\max_{z} w^T A z$

$\Sigma w = 1$

$\Sigma z_i = 1$

$w \geq 0$

$z \geq 0$

1. The optimization problem with relaxed constraints can be solved with algorithms (Dantzig)
   - $\max_z \min_w f(w, z) = \min_w \max_z f(w, z)$
2. Since $f(w, z)$ is convex w.r.t. w, and concave w.r.t. z, the solution can be reduced to constraint I (row and column selection), $(\tilde{w}, \tilde{z})$.
3. From 2, $(\tilde{w}, \tilde{z})$ is a saddle point by definition.

Geometric Interpretation

$min\, f_o(x)$ $\quad (t)$
$s.t.\, f_1(x) \le 0$ $\quad (u \le 0)$

$L(x, \lambda) = \underset{t}{f_o(x)} + \lambda \underset{u}{f_1(x)}$

$g(\lambda) = \min_{(u,t)\in G} t + \lambda u \qquad G = \{(\overset{u}{f_1(x)}, \overset{t}{f_o(x)}) | x \in D\}$

$L(x,\lambda) = t + \lambda u$

$g(\lambda) = \min_x L(x,\lambda)$

$= \min_x t + \lambda u$

$= -\max_x -t - \lambda u$

$= -\max_x \underbrace{(-\lambda)}_{y} u - t$

$g(\lambda) = \lambda u + t$
supporting hyperplane to G
that intersects t axis at $t = g(\lambda)$

u

Duality via Separating Hyperplane

Set $G = \{\left(f_1(x), \ldots, f_m(x), h_1(x), \ldots, h_p(x), f_o(x)\right) | x \in D\}$,
$G \in R^m \times R^p \times R, p^* = \inf\{t | (u, w, t) \in g, u \le 0, w = 0\}$

Lagrangian $L = (\lambda, v, 1)^T (u, w, t) = \sum_{i=1}^m \lambda_i u_i + \sum_{i=1}^p v_i w_i + t$
Dual Problem $g(\lambda, v) = \inf\{(\lambda, v, 1)^T (u, w, t) | (u, w, t) \in G\}$

Separating hyperplane: Example

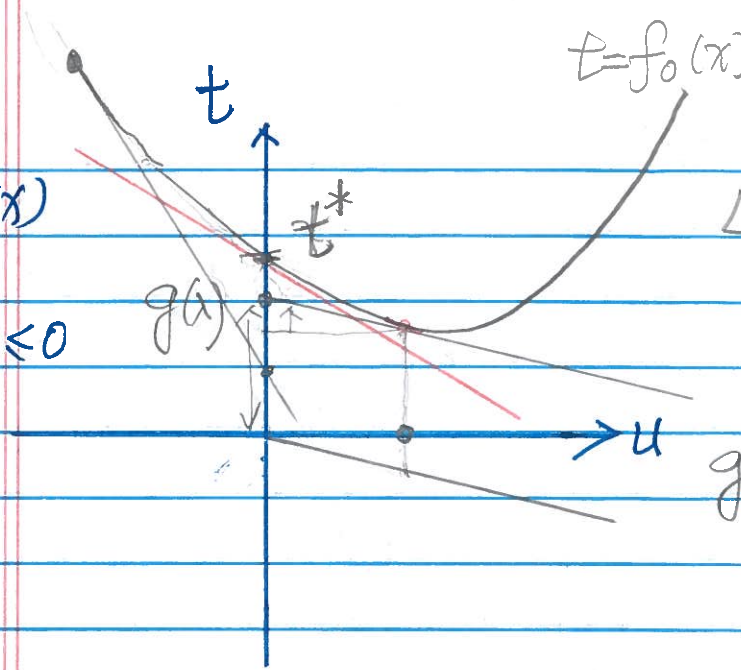
$$\{(u,t) | f_o(x) \le t, f_1(x) \le u, \exists x \in D\}$$

$$(\tilde{\lambda}, \tilde{v}, \tilde{\mu})^T (u, w, t) \ge \alpha, \qquad \forall (u, w, t) \in A$$
$$(\tilde{\lambda}, \tilde{v}, \tilde{\mu})^T (u, w, t) \le \alpha, \qquad \forall (u, w, t) \in B$$

Since $\tilde{\mu} \ne 0$, we can have $(\lambda, v, 1) = (\frac{\tilde{\lambda}}{\tilde{\mu}}, \frac{\tilde{v}}{\tilde{\mu}}, 1)$

$$A = \{(u, w, t) | \exists x \in D, f_i(x) \le u_i, i = 1, \ldots, m,$$
$$h_i(x) = w_i, i = 1, \ldots, p, f_o(x) \le t\}$$

$\min f_0(x)$
const
$\lambda\ f_1(x) \le 0$



$\{(f_1(x), f_0(x)) \mid x \in D\}$

$$L(x,\lambda) = f_0(x) + \lambda f_1(x) \quad (\lambda \ge 0)$$

$$= f_0(x) - (-\lambda) f_1(x)$$

$$= t - (-\lambda) u$$

$$g(\lambda) = \min_x L(x,\lambda) \qquad (\bar{x}) \text{ is a sol.}$$

$$= f_0(\bar{x}) - (-\lambda) f_1(\bar{x})$$

$$\left(\nabla_u f_0(\bar{x}) = -\lambda \right)$$

$$\max_\lambda g(\lambda)$$

Supporting Hyperplane

$$t = f_0(\bar{x}) + \nabla_u f_0(\bar{x})(u - \bar{u})$$

$$= t - (-\lambda)(u - \bar{u})$$

Dual $\max_\lambda g(\lambda)$

$= g(\lambda^*)$ when

$\nabla_u f_0(x) = -\lambda^*$

at $u^* = 0$.

$f_0(x) = x^2 - 4x + 2$, $(x)u \le 0$. $L(\lambda, x) = f_0(x) + \lambda x$

$g(\lambda) = \min_x L(\lambda, x)$

$f^*(y) = \max_x L(x, y) = yx - (x^2 - 4x + 2)$

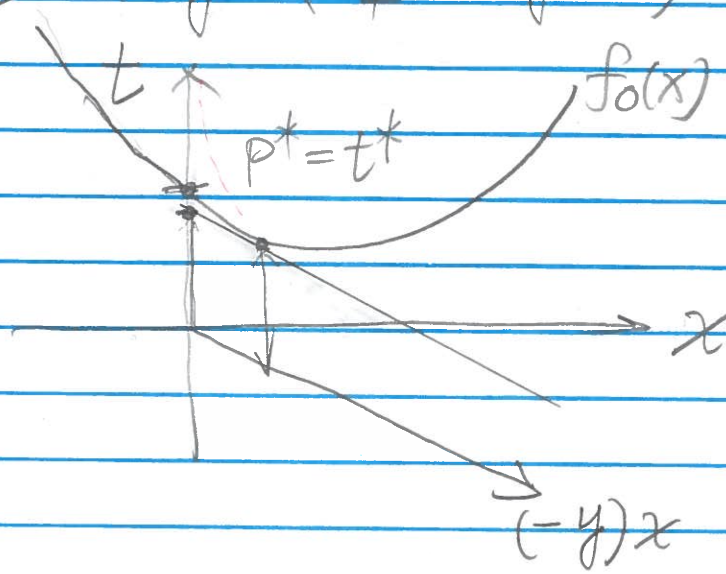
$\nabla_x L(x,y) = y - 2x + 4 = 0$

$= y^2/4 + 2y + 2$

$x = \frac{y}{2} + 2$

Supporting Hyperplane

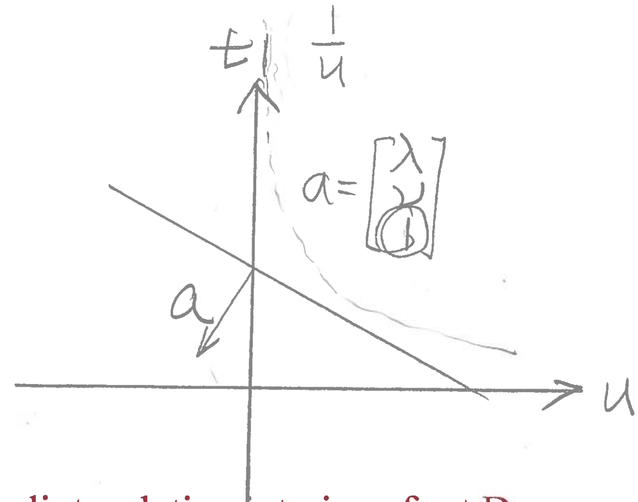
$t = xy - (\frac{y^2}{4} + 2y + 2)$



$g(x^*) = \max_\lambda g(\lambda)$

Lagrange dual problem

$$\max g(\lambda, v)$$
$$s.t. \quad \lambda \geq 0$$



Properties
This is a convex problem.
The opt. solution is denoted as d^*
$p^* - d^* = gap \geq 0$
If $gap > 0$, it is a weak duality.
If $gap = 0$, it is a strong duality.

relint: relative interior of set D

Slater's condition
Given that the primal problem is convex,

If $f_i(x) < 0, i = 1, \ldots, m, \exists x \in relint\ D$
Then strong duality holds.

$$relint\ C = \{x \in C | B(x, r) \cap aff\ C \subseteq C \ for\ some\ r > 0\}$$
$$B(x, r) = \{y | \|y - x\| \leq r\}$$

any norm

Shadow Price Interpretation: Food vs. Vitamin

Flour	protein powder
Veg.	vitamins A,B,D,E,K
Fruits	minerals

Primal $\min c^T x \quad s.t.\ Ax \succcurlyeq b, \quad x \succcurlyeq 0$

$\min c^T x \quad s.t.\ -Ax + b \preccurlyeq 0, \quad -x \preccurlyeq 0$

$$\min c_1 x_1 + c_2 x_2 + c_3 x_3 \quad \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \succcurlyeq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, x_i \geq 0, \forall i$$

Dual $\max \lambda^T b \quad s.t.\ A^T \lambda \preccurlyeq c, \quad \lambda \succcurlyeq 0$

$$\max \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$
$$\text{s.t.} \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \preccurlyeq \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Lagrangian $L(x, \lambda) = c^T x + \lambda_1^T(-Ax + b) + \lambda_2^T(-x)$

$$= [c^T + \lambda_1^T(-A) - \lambda_2^T]x + \lambda_1^T b$$
$$c^T = \lambda_1^T(A) + \lambda_2^T, \ or\ A^T \lambda_1 \leq c$$

Shadow Price Interpretation: Spring Energy & Force

$$min\, f_o(x_1, x_2) = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3(l - x_2)^2$$

f_o : potential energy $k_i > 0$: stiffness constant of spring i

$w/2 - x_1 \leq 0$

$w + x_1 - x_2 \leq 0$

$w/2 - l + x_2 \leq 0$

$$min \frac{1}{2}(k_1x_1^2 + k_2(x_2 - x_1)^2 + k_3(l - x_2)^2)$$

$\lambda_1 \quad w/2 - x_1 \leq 0$

$\lambda_2 \quad w + x_1 - x_2 \leq 0$

$\lambda_3 \quad w/2 - l + x_2 \leq 0$

$\lambda_1(w/2 - x_1) = 0, \lambda_2(w - x_2 + x_1) = 0, \lambda_3(w/2 - l + x_2) = 0$

zero gradient condition

$$\begin{bmatrix} k_1x_1 - k_2(x_2 - x_1) \\ k_2(x_2 - x_1) - k_3(l - x_2) \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

λ_i : contact forces between the walls & blocks

KKT (Karush-Kuhn-Tucker) Conditions

2. $f_i(x), i = 1, \dots, m,\ \ h_i(x),\ \ i = 1, \dots, p\ are\ differentiable$

1. Primal constraints : $f_i(x) \leq 0, i = 1, \dots, m.$
   $h_i(x) = 0, i = 1, \dots, p.$

2. Dual constraints : $\lambda \succcurlyeq 0$

3. Complementary slackness : $\lambda_i f_i(x) = 0, i = 1, \dots, m.$

4. Gradient of Lagrangian with respect to x variables
   $\nabla_x f_o(x) + \sum_{i=1}^{m} \lambda_i \nabla_x f_i(x) + \sum_{i=1}^{p} v_i \nabla_x h_i(x) = 0$