Interpretation: Saddle-point

$\min_{x} \max_{\lambda,\nu}$ $\max_{\lambda,\nu} \min_{x} L(x,\lambda,\nu)$ $f(\omega, z)$ $\min_{\omega} \max_{z}$

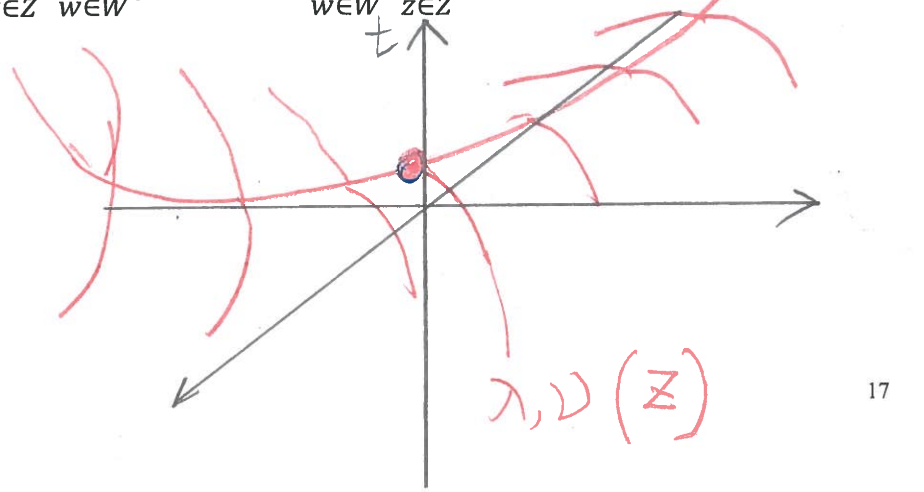
Claim : Result of II $\geq$ Result of I

Given an arbitrary pair $(\widetilde{w}, \tilde{z}) \in D$

$$\min_{w \in W} f(w, \tilde{z}) \leq f(\widetilde{w}, \tilde{z}) \leq \max_{z \in Z} f(\widetilde{w}, z) \quad \forall \widetilde{w}, \tilde{z} \in D$$

$$\min_{w \in W} f(w, \tilde{z}) \leq \max_{z \in Z} f(\widetilde{w}, z)$$

Thus $\max_{z \in Z} \min_{w \in W} f(w, z) \leq \min_{w \in W} \max_{z \in Z} f(w, z)$



Interpretation: Saddle-point

Example : $f(w,z)$ $w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

$z = 1, \quad 2, \quad 3$

z	w
$\min_{w \in W} f(w, 1) = 1$	$\max_{z \in Z} f(1, z) = 1$
$\min_{w \in W} f(w, 2) = 1$	$\max_{z \in Z} f(2, z) = 2$
$\min_{w \in W} f(w, 3) = 1$	$\max_{z \in Z} f(3, z) = 3$
$\max_{z \in Z} \min_{w \in W} f(w, z) = 1$	$\min_{w \in W} \max_{z \in Z} f(w, z) = 1$

Interpretation: Saddle-point

$$\text{Example}: f(w,z)\quad w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$
$$z = 1,\quad 2,\quad 3$$

$$\min_{w\in W} f(w, \overset{z}{1}) = 1 \qquad \max_{z\in Z} f(\overset{w}{1}, z) = 3$$
$$\min_{w\in W} f(w,2) = 1 \qquad \max_{z\in Z} f(2,z) = 3$$
$$\min_{w\in W} f(w,3) = 1 \qquad \max_{z\in Z} f(3,z) = 3$$

$$\max_{z\in Z}\min_{w\in W} f(w,z) = 1 \qquad \min_{w\in W}\max_{z\in Z} f(w,z) = 3$$

Interpretation: Saddle-point

$$\text{Example}: f(w,z)\quad w = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 4 & 6 & 3 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$$
$$z = 1,\quad 2,\quad 3$$

convex w.r.t w

concave w.r.t. z

$$\min f(w,1) = 2 \qquad \max f(1,z) = 6$$
$$f(w,2) = 3 \qquad f(2,z) = 3$$
$$f(w,3) = 1 \qquad f(3,z) = 5$$

$$\max_{z}\min_{w} f(w,z) = 3 \qquad \min_{w}\max_{z} f(w,z) = 3$$
$$(2,2) \qquad (2,2)$$

Saddle Point Von Neumann, Dantzig, Nash

1. Definition

Given function $f(w, z)$,

$(\tilde{w}, \tilde{z})$ is a saddle point of $f(w, z)$

if $\max_{z} f(\tilde{w}, z) = f(\tilde{w}, \tilde{z})$

$min_w f(w, \tilde{z}) = f(\tilde{w}, \tilde{z})$

2. Theorem I

$\max_{z} \min_{w} f(w, z) = \min_{w} \max_{z} f(w, z)$

iff a saddle point of $f(w, z)$ exists.

Saddle Point: Theorem I Proof

$\max_{z} \min_{w} f(w, z) = \min_{w} \max_{z} f(w, z)$

iff a saddle point of $f(w, z)$ exists.

Proof: => Let

$\tilde{w} = \arg_w \min_{w} \max_{z} f(w, z)$

$\tilde{z} = \arg_z \max_z \min_w f(w, z)$

We have

$f(\tilde{w}, \tilde{z}) \leq \max_{z} f(\tilde{w}, z) = \min_{w} f(w, \tilde{z}) \leq f(\tilde{w}, \tilde{z})$

By definition $(\tilde{w}, \tilde{z})$ is a saddle point.

Saddle Point: Theorem I Proof

$$\max_z \min_w f(w,z) = \min_w \max_z f(w,z)$$

iff a saddle point of $f(w,z)$ exists.

Proof: <= Assume that $(\widetilde{w}, \tilde{z})$ is a saddle point.

We have

$$\max_z \min_w f(w,z) \geq \min_w f(w,\tilde{z}) = f(\widetilde{w},\tilde{z})$$

$$\min_w \max_z f(w,z) \leq \max_z f(\widetilde{w},z) = f(\widetilde{w},\tilde{z})$$

Thus, $\max_z \min_w f(w,z) = \min_w \max_z f(w,z)$

Saddle Point: 2-D Table Formulation

The row and column selection is formulated as a bilinear optimization problem

- $f(w,z) = \sum_i \sum_j a_{ij}\, w_i z_j$

Constraint I: row and column selection

- $w_i, z_j \in \{0,1\}, \sum_i w_i = 1, \sum_j z_j = 1.$

Constraint II: Relaxed discrete constraints

$\sum_i w_i = 1, \sum_j z_j = 1,\ w_i \geq 0, z_j \geq 0, \forall i,j$

Saddle Point: 2-D Table Formulation

Find a saddle point of $f(w, z)$ under constraint I.

Theorem II: A saddle point of 2-D table formulation can be solved if $f(w, z)$ is convex w.r.t. w, and concave w.r.t. z.

Saddle Point:

1. The optimization problem with relaxed constraints can be solved with algorithms (Dantzig)
   - $\max_z \min_w f(w, z) = \min_w \max_z f(w, z)$

2. Since $f(w, z)$ is convex w.r.t. w, and concave w.r.t. z, the solution can be reduced to constraint I (row and column selection), $(\widetilde{w}, \tilde{z})$.

3. From 2, $(\widetilde{w}, \tilde{z})$ is a saddle point by definition.

$\min_{w} w^T A z$

$\max_{z} w^T A z$

$\Sigma w = 1$

$\Sigma z_i = 1$

$w \geq 0$

$z \geq 0$

Geometric Interpretation

$min\, f_o(x) \qquad (t)$
$s.t.\, f_1(x) \leq 0 \;\; (u \leq 0)$
$g(\lambda) = \min_{(u,t)\in G} t + \lambda u \qquad G = \{(f_1(x), f_o(x)) | x \in D\}$

$g(\lambda) = \lambda u + t$
supporting hyperplane to G
that intersects t axis at $t = g(\lambda)$

u

Duality via Separating Hyperplane

Set $G = \{\left(f_1(x), \dots, f_m(x), h_1(x), \dots, h_p(x), f_o(x)\right) | \mathrm{x} \in D\}$,
$G \in R^m \times R^p \times R, p^* = \inf\{t | (u, w, t) \in g, u \leq 0, w = 0\}$

Lagrangian $L = (\lambda, v, 1)^T(u, w, t) = \sum_{i=1}^{m} \lambda_i u_i + \sum_{i=1}^{p} v_i w_i + t$
Dual Problem $g(\lambda, v) = \inf\{(\lambda, v, 1)^T(u, w, t) | (u, w, t) \in G\}$

Separating hyperplane: Example

$$\{(u, t) | f_o(x) \leq t, f_1(x) \leq u, \exists x \in D\}$$

$$\left(\tilde{\lambda}, \tilde{v}, \tilde{\mu}\right)^T (u, w, t) \geq \alpha, \qquad \forall (u, w, t) \in A$$
$$\left(\tilde{\lambda}, \tilde{v}, \tilde{\mu}\right)^T (u, w, t) \leq \alpha, \qquad \forall (u, w, t) \in B$$

Since $\tilde{\mu} \neq 0$, we can have $(\lambda, v, 1) = (\frac{\tilde{\lambda}}{\tilde{\mu}}, \frac{\tilde{v}}{\tilde{\mu}}, 1)$

$$A = \{(u, w, t) | \exists x \in D, f_i(x) \leq u_i, i = 1, \dots, m,$$
$$h_i(x) = w_i, i = 1, \dots, p, f_o(x) \leq t\}$$