

CSE203B Convex Optimization: Chapter 4: Problem Statement

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Convex Optimization Formulation

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1. Introduction

Formulation: One of the most critical processes to conduct a project.

$$\begin{aligned} \min & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \quad (Ax = b \text{ Affine set}) \end{aligned}$$

$$x \in R^n$$

$$D_{f_0} f_0: R^n \rightarrow R$$

$$D_{f_i} f_i: R^n \rightarrow R$$

$$D_{h_i} h_i: R^n \rightarrow R$$

f_0, f_i, \dots, f_m are convex

$D = \bigcap_{i=0,m} D_{f_i} \cap \bigcap_{i=0,p} D_{h_i}$ **Domain of functions**, but not the feasible set.

Feasible Set: The set which satisfies the constraints (is convex for convex problems).

1.1 Introduction: Eliminating Equality Constraints

$$\begin{aligned} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & Ax = b \end{aligned}$$

a. Convert $\{x | Ax = b\}$ to $\{Fz + x_0 | z \in R^k\}$

b. We have an equivalent problem

$$\begin{aligned} \min & f_0(Fz + x_0) \\ \text{s.t.} & f_i(Fz + x_0) \leq 0 \end{aligned}$$

Remark: Matrix F contains columns of null space basis

1.2 Introduction: Slack Variables

$$\begin{aligned} \min & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0, i = 1, \dots, m \\ & Ax = b \end{aligned}$$

Add slack variables to convert to an equivalent problem

a. Convert the objective function with variable t

$$\begin{aligned} \min & t \\ \text{s. t.} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, i = 1, \dots, m \\ & A^T x = b \end{aligned}$$

b. Convert the inequality with variables s_i

$$\begin{aligned} \min & f_0(x) \\ \text{s. t.} & f_i(x) + s_i = 0 \\ & A^T x = b \\ & s_i \in R_+, i = 1, \dots, m \end{aligned}$$

1.3 Introduction: Absolute values and Softmax

a. Absolute values

$$|f_i(x)| \leq b$$

$$\Rightarrow f_i(x) \leq b \text{ and}$$

$$-f_i(x) \leq b$$

b. Maximum values

$$\max\{f_1, f_2, \dots, f_m\}$$

$$\text{Softmax: } \frac{1}{\alpha} \log (e^{\alpha f_1} + e^{\alpha f_2} + \dots + e^{\alpha f_m})$$

Example: $\max\{1, 5, 10, 2, 3\} \Rightarrow \text{Softmax}$

$$\frac{1}{\alpha} \log(e^{\alpha} + e^{5\alpha} + e^{10\alpha} + e^{2\alpha} + e^{3\alpha}) \approx 10$$

2.1 Optimality Conditions: Local vs. Global Optima

Definition: Local Optima

Given a convex optimization problem and a point $\bar{x} \in R^n$

If there exists a $r > 0$

s. t. $f_0(z) \geq f_0(\bar{x})$ for all $z \in \text{Feasible Set}$, and $\|z - \bar{x}\|_2 \leq r$

Then \bar{x} is a local optimum.

2.2 Optimality Conditions

Theorem: Given a convex opt. problem

If \bar{x} is a local optimum, then \bar{x} is a global optimum

Proof: By contradiction

Suppose that $\exists y \in \text{Feasible Set}$

$$\text{s.t. } f_0(\bar{x}) > f_0(y)$$

We have $f_0(\bar{x}) > (1 - \theta)f_0(\bar{x}) + \theta f_0(\bar{y})$ (*by assumption*)
 $> f_0((1 - \theta)\bar{x} + \theta\bar{y})$ (*f_0 is convex*)

And $(1 - \theta)\bar{x} + \theta\bar{y}$ is feasible (**Feasible set is convex**)

The inequality contradicts to the assumption of local optima.

2.2 Optimality Criterion for Differentiable $f_0(x)$

Theorem: If $\nabla f_0(x)^T(y - x) \geq 0$, for a given $x \in \text{Feasible Set}$ and for all $y \in \text{Feasible Set}$, then x is optimal.

(i. e. $K = \{y - x | y \in \text{feasible set}\}, \nabla f_0(x) \in K^*$)

Proof: From the first order condition of convex function, we have $f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y - x)$.

Given the condition that $\nabla f_0^T(x)(y - x) \geq 0, \forall y$ in feasible set. We have $f_0(y) \geq f_0(x), \forall y$ in feasible set, which implies that x is optimal.

Remark: $\nabla f_0^T(x)(y - x) = 0$ is a supporting hyperplane to feasible set at x .

2.2.1 Optimality Criterion without Constraints

Theorem: For problem $\min f_0(x), x \in R^n$, where f_0 is convex, the optimal condition is $\nabla f_0(x) = 0$.

Proof: ($\nabla f_0(x) = 0 \Rightarrow$ Optimality)

Since $f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x), \forall x, y \in R^n$ (first order condition of convex function)

We have $f_0(y) \geq f_0(x)$.

Therefore, x is an optimal solution.

($\nabla f_0(x) = 0 \Leftarrow$ Optimality) By contradiction

2.2.2 Opt. with Inequality Constraints

Problem: Min $f_0(x)$

$$s.t. Ax \leq b, A \in R^{m \times n}$$

Suppose that $A\bar{x} = b$ (one particular case).

Let $x = \bar{x} + u$.

We can write $\begin{cases} \min f_0(\bar{x} + u) \\ Au \leq 0 \end{cases}$

Opt. condition: $\nabla f_0(x)^T u \geq 0, \forall \{u | Au \leq 0\} \equiv K$

In other words,

$$\nabla f_0(\bar{x}) \in K^* \text{ of } K = \{u | Au \leq 0\} \text{ and } K^* = \{-A^T v | v \geq 0\}$$

$$i.e. \nabla f_0(\bar{x}) = -A^T v, \exists v \in R_+^m$$

$$\nabla f_0(\bar{x}) + A^T v = 0, v \geq 0.$$

2.2.3 Opt. with Equality Constraints

$$\begin{cases} \min f_0(x) \\ \text{s.t. } Ax = b \end{cases}$$

Let $x = \bar{x} + u$ and $A\bar{x} = b$,

$$\text{we have } \begin{cases} \min f_0(\bar{x} + u) \\ Au = 0 \end{cases}, K = \{u | Au = 0\}$$
$$\nabla f_0(\bar{x}) \in K^*, K^* = \{A^T v | v \in R^p\}$$
$$\nabla f_0(\bar{x}) + A^T v = 0$$

$$\text{Let } K_1 = \{u | Au \geq 0\}$$

$$K_2 = \{u | -Au \geq 0\}$$

$$K_1 \cap K_2 = \{u | Au \geq 0, -Au \geq 0\}$$

We have

$$\begin{aligned} (K_1 \cap K_2)^* &= \{A^T v_1 + (-A)^T v_2 | v_1, v_2 \geq 0\} \\ &= \{A^T v | v \in R^p\} \end{aligned}$$

2.2.3 Opt. with Equality Constraints: Example

$$\begin{aligned} \min_x f(x) &= x_1^2 + x_2^2 \\ \text{s.t. } [2 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 3 \end{aligned}$$

We can derive $x^* = (x_1^*, x_2^*) = \left(\frac{6}{5}, \frac{3}{5}\right)$

$$\nabla f(x^*) = \begin{bmatrix} 2x_1^* \\ 2x_2^* \end{bmatrix} = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix}, \quad \nabla f(x^*) + A^T v = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \times \left(-\frac{6}{5}\right) = 0$$

New Problem:

$$\begin{aligned} \nabla f(x) + A^T v &= 0 \\ Ax &= b \end{aligned} \Rightarrow \begin{aligned} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} v &= 0 \\ [2 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 3 \end{aligned}$$

2.3 Quasiconvex Functions

$f: R^n \rightarrow R$ is called quasiconvex (unimodal)

sublevel set $S_t = \{x | x \in \text{dom } f, f(x) \leq t\}$

if its domain and all sublevel sets $S_t, \forall t \in R$ are convex,

$f: R^n \rightarrow R$ is called quasiconcave if $-f$ is quasiconvex.

$f(x)$ quasiconvex and quasiconcave \rightarrow quasilinear

Ex: $\log x, x \in R_{++}$

2.3 Quasiconvex Functions

Ex: Ceiling function

$$\text{Ceil}(x) = \inf\{z \in Z \mid z > x\} : \text{quasilinear}$$

$$\text{Ex: } f(x_1, x_2) = x_1 x_2 = \frac{1}{2} [x_1 \quad x_2] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is quasiconcave in R_+^2 , $S_t = \{x \in R_+^2 \mid x_1 x_2 \geq t\}$

$$\text{Ex: } f(x) = \frac{a^T x + b}{c^T x + d} \text{ for } c^T x + d > 0$$

$$S_t = \{x \mid c^T x + d > 0, a^T x + b \leq t(c^T x + d)\}$$

open halfspace closed halfspace

→ S_t is convex (t is given here)

→ $f(x)$ is $\left. \begin{array}{l} \text{quasiconvex} \\ \text{quasiconcave} \end{array} \right\} \rightarrow \text{quasilinear}$

2.3 Quasiconvex Optimization

$\min f_0(x)$ ($f_0(x)$ is **quasiconvex**, f_i 's are convex.)

s. t. $f_i(x) \leq 0, i = 1, \dots, m$

$$Ax = b$$

Remark: A locally opt. solution $(x, f_0(x))$ may not be globally opt.

Algorithm: Bisection method for quasiconvex optimization.

Given $l \leq p^* \leq u, \epsilon > 0$

Repeat 1. $t = (l + u)/2$

Find a

convex function

2. Find a feasible solution x :

$$\text{s. t. } \Phi_t(x) \leq 0 \quad (f_0(x) \leq t \Leftrightarrow \Phi_t(x) \leq 0)$$

$$f_i(x) \leq 0$$

$$Ax = b$$

3. If solution is feasible, $u = t$, else $l = t$

Until $u - l \leq \epsilon$

Ex: $f(x) = \frac{p(x)}{q(x)} \leq t \rightarrow p(x) - tq(x) \leq 0$ (p is convex & q is

concave)

3. Linear Programming: Format

General Form :

$$\begin{aligned} & \min c^T x \\ & s. t. \quad Gx \leq h, \quad G \in R^{m*n}, A \in R^{p*n} \\ & \quad \quad Ax = b \end{aligned}$$

Standard Form :

$$\begin{aligned} & \min c^T x \\ & s. t. \quad Ax = b \\ & \quad \quad x \geq 0 \end{aligned}$$

Remark: Figure out three possible situations

1. No feasible solutions
2. Unbounded solutions
3. Bounded solutions

3. Linear Programming: Cases

$$\min c^T x$$

$$\text{s.t. } Ax = b$$

(1) No feasible solutions: $b \notin R(A)$ (b is not in the range of A)

$$\text{e.g. } \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

(2) Unbounded solutions: $b \in R(A)$ but $c \notin R(A^T)$

$$\text{e.g. } \min [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[1 \quad 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \quad (\text{The solution } \rightarrow -\infty)$$

(3) Bounded solutions: $b \in R(A), c \in R(A^T)$

$$\text{e.g. } \min [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\text{Thus } x^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, f(x^*) = [1 \quad 1] \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2$$

3. Linear Fractional Programming

$$\begin{aligned} \text{P1: } \min f_o(x) &= \frac{c^T x + d}{e^T x + f}, \quad \text{dom } f_o = \{x \mid e^T x + f > 0\} \\ \text{s. t. } Gx &\leq h \\ Ax &= b \end{aligned}$$

$$\text{P1} \Rightarrow \text{P2: } \text{Let } y = \frac{x}{e^T x + f}, \quad z = \frac{1}{e^T x + f}$$

$$\begin{aligned} \text{P2: } \min c^T y + dz \\ \text{s. t. } Gy - hz &\leq 0 \\ Ay - bz &= 0 \\ e^T y + fz &= 1 \\ z &\geq 0 \end{aligned}$$

4. Quadratic Opt. Problems (QP)

$$\text{QP} : \min \frac{1}{2} x^T P x + q^T x + r$$

$$s. t. \quad Gx \preceq h$$

$$Ax = b$$

$$P \in S_+^n, \quad G \in R^{m \times n}, \quad A \in R^{p \times n}$$

QCQP : (Quadratically Constrained Quadratic Program)

$$\min \frac{1}{2} x^T P_0 x + q_0^T x + r_0$$

$$s. t. \quad \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \dots, m$$

$$Ax = b$$

$$P_i \in S_+^n, \quad i = 0, 1, \dots, m$$

4. Quadratic Opt. Problems (SOCP)

SOCP : (Second-Order Cone Program)

$$\min f^T x$$

$$\text{s. t. } \|A_i x + b_i\|_2 \leq c_i^T x + d_i, i = 1, \dots, m$$

$$Fx = g$$

SOCP: $(Ax + b, c^T x + d)$ lies in the second order cone

$$\{(y, t) \mid \|y\|_2 \leq t, y \in R^k\}$$

QCQP viewed as SOCP

QCQP constraint: $x^T A^T A x + b^T x + c \leq 0$

can be expressed as a SOCP constraint:

$$\left\| \begin{array}{c} 1 + b^T x + c \\ 2 \\ Ax \end{array} \right\|_2 \leq (1 - b^T x - c)/2$$

4. Quadratic Opt. Problems (SOCP)

SOCP : (Second-Order Cone Program)

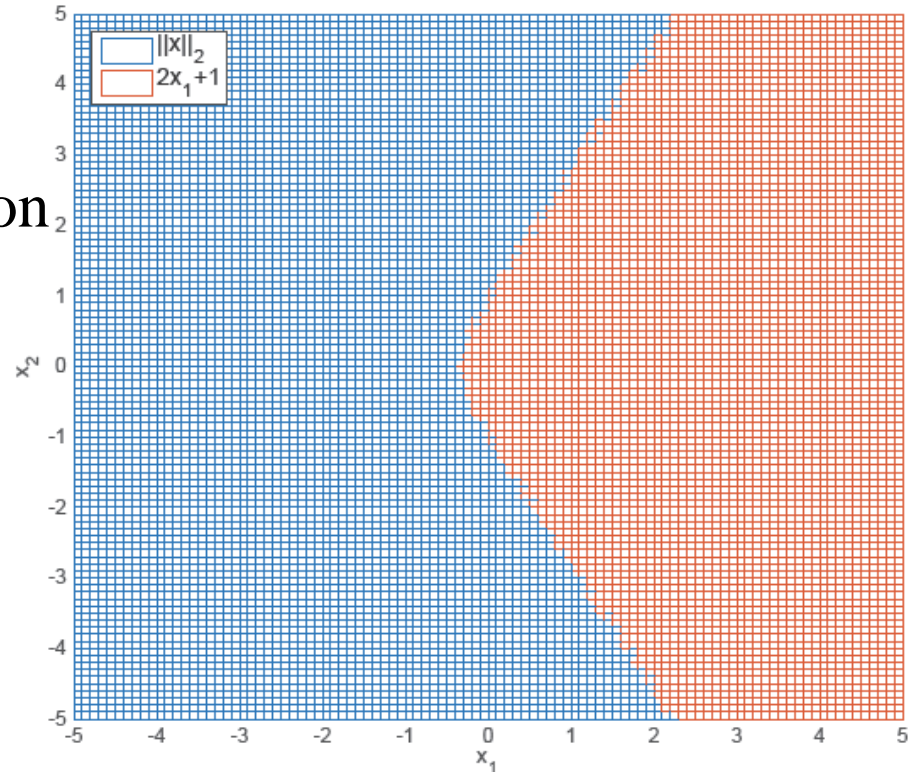
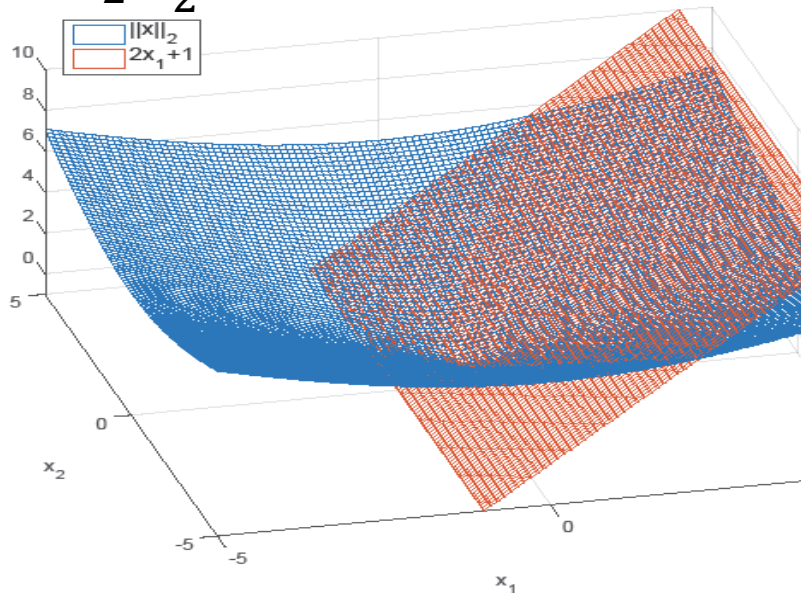
$$\min f^T x$$

$$s. t. \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i, i = 1, \dots, m$$

$$F x = g$$

Example: SOCP constraint:

$$\left\| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\|_2 \leq 2x_1 + 1, \text{ feasible region}$$



4. Quadratic Opt. Problems (SOCP)

SOCP : (Second-Order Cone Program)

$$\min f^T x$$

$$s. t. \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i, i = 1, \dots, m$$

$$F x = g$$

SOCP: $(Ax + b, c^T x + d)$ lies in the second order cone

$$\{(y, t) \mid \|y\|_2 \leq t, y \in R^k\}$$

SOCP viewed as a Semidefinite Program Problem

$$\text{SOCP constraint: } \|Ax + b\|_2 \leq c^T x + d$$

can be expressed as a Semidefinite Program constraint:

$$\begin{bmatrix} (c^T x + d)I & Ax + b \\ (Ax + b)^T & c^T x + d \end{bmatrix} \succcurlyeq 0$$

5. Geometric Programming

$$f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}, \quad c_k > 0, a_{ik} \in R, x \in R_{++}^n$$

Each term is called monomial

$f(x)$ is called posynomial

Geometric Program:

$$\min f_o(x)$$

s.t.

$$f_i(x) \leq 1, i = 1, \dots, m$$

$$h_i(x) = 1, i = 1, \dots, p$$

$$x > 0$$

f_i s are posynomials

h_i s are monomials

5. Geometric programming in convex form

monomial $f(x) = cx_1^{a_1} \dots x_n^{a_n}$, $x \in R_{++}^n$

$$\log f(e^{y_1}, \dots, e^{y_n}) = a^T y + b, \quad b = \log c$$

polynomial $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} \dots x_n^{a_{nk}}$

$$\log f(e^{y_1} \dots e^{y_n}) = \log \sum_{k=1}^K e^{a_k^T y + b_k}, \quad b_k = \log c_k$$

Geometric program transform

$$\min \log \left(\sum_{k=1}^{K_0} e^{a_{ok}^T y + b_{ok}} \right)$$

$$\text{subject to } \log \sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \leq 0, \quad i = 1, \dots, m$$

$$Gy + d = 0$$

6. Generalized Inequality Constraints

$$\begin{aligned} \min & f_0(x) \\ \text{s. t.} & f_i(x) \preceq_{K_i} 0 \\ & Ax = b \\ & (x \preceq_K y \rightarrow y - x \in K) \end{aligned}$$

Semidefinite Programming (SDP)

$$\begin{aligned} \min & c^T x \\ \text{s. t.} & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \\ & G, F_1, \dots, F_n \in S^k, A \in R^{p \times n} \end{aligned}$$

Standard Form SDP

$$\begin{aligned} \min & \text{tr}(CX) \\ \text{s. t.} & \text{tr}(A_i X) = b_i, \quad i = 1, \dots, p \\ & X \succeq 0 \\ & C, A_1, \dots, A_p \in S^n, X \in S^n \end{aligned}$$

Summary

(1). $LP \subset QP \subset QCQP \subset SOCP \subset SDP$

(2). Software Tools (Examples)

CVX: Matlab software for disciplined convex (Boyd)

CPLEX: IP, LP, QP, SOCP (IBM)

Gurobi: LP, QP, MILP, MIQP, MIQCP (Gu, Rothberg, Bixby)

(3). Check if the problem is convex

Summary

- (1). Format of the formulation
 - a. Follow the format of the solver (software package)
 - b. Find equivalent formulation for simpler approaches (coding, complexity, accuracy)
- (2). Feasibility of the solution
Check if the feasible set is not empty.
- (3). Boundness of the solution
Check if the solution is bounded (reasonable, not $-\infty$)
- (4). Optimality of the solution
Check the supporting hyperplane of object function