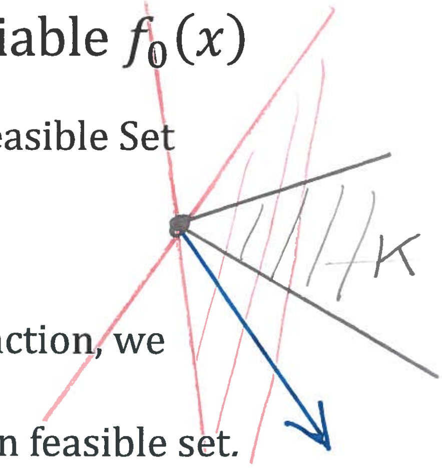
2.2 Optimality Criterion for Differentiable $f_0(x)$

Theorem: If $\nabla f_0(\bar{x})^T(y-\bar{x}) \geq 0$, for a given $\bar{x}$ ∈Feasible Set and for all $y \in$ Feasible Set, then $\bar{x}$ is optimal.
(i.e. $K = \{y - \bar{x} | y \in feasible\ set\}, \nabla f_0(\bar{x}) \in K^*$)



Proof: From the first order condition of convex function, we have $f_0(y) \geq f_0(\bar{x}) + \nabla f_0(\bar{x})^T(y-\bar{x})$.
Given the condition that $\nabla f_0^T(\bar{x})(y-\bar{x}) \geq 0,\ \forall y$ in feasible set.
We have $f_0(y) \geq f_0(\bar{x}),\ \forall y$ in feasible set, which implies that $\bar{x}$ is optimal.

$K = \{\sum_i u_i \theta_i \mid \theta \geq 0\}$

Remark: $\nabla f_0^T(x)(y-x) = 0$ is a supporting hyperplane to feasible set at x.

$f_0(x) \geq f_0(\bar{x}) + \nabla f_0(\bar{x})^T(x-\bar{x})$

Supporting Hyperplane $t = f_0(\bar{x}) + \nabla f_0(\bar{x})^T(x-\bar{x})$

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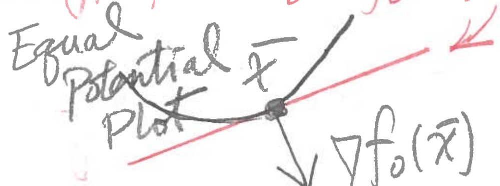


2.2.1 Optimality Criterion without Constraints

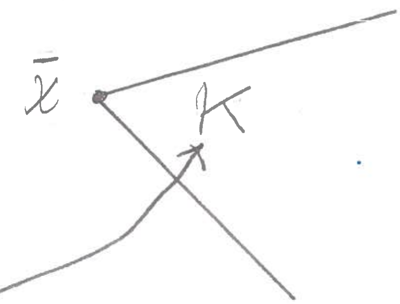
Theorem: For problem min $f_0(x), x \in R^n$, where f_0 is convex, the optimal condition is $\nabla f_0(x) = 0$.

Proof: ($\nabla f_0(x) = 0 \Rightarrow$ Optimality)
Since $f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x), \forall x, y \in R^n$ (first order condition of convex function)
We have $f_0(y) \geq f_0(x)$.
Therefore, x is an optimal solution.
($\nabla f_0(x) = 0 \Leftarrow$ Optimality) By contradiction

2.2.2 Opt. with Inequality Constraints

Problem: Min $f_0(x)$

$$s.t.\ Ax \leq b, A \in R^{m\times n}$$



Suppose that $A\bar{x} = b$ (one particular case).

Let $x = \bar{x} + u$.

We can write $\begin{cases} \min f_0(\bar{x}+u) \\ \quad Au \leq 0 \end{cases}$

Opt. condition: $\nabla f_0(\bar{x})^T u \geq 0,\ \forall\{u|Au \leq 0\} \equiv K$

In other words,

$$\nabla f_0(\bar{x}) \in K^*\ of\ K = \{u|Au \leq 0\}\ and\ K^* = \{-A^T v|v \geq 0\}$$

i.e. $\nabla f_0(\bar{x}) = -A^T v, \exists v \in R_+^m$

$$\nabla f_0(\bar{x}) + A^T v = 0,\ v \geq 0.$$

2.2.3 Opt. with Equality Constraints

$$\begin{cases} \min f_0(x) \\ s.t.\ Ax = b \end{cases}$$

Let $x = \bar{x} + u$ and $A\bar{x} = b$,

we have $\begin{cases} \min f_0(\bar{x}+u) \\ \quad Au = 0 \end{cases}$, $K = \{u|Au = 0\}$ $\quad Au=0$

$-Au=0$

$$\nabla f_0(\bar{x}) \in K^*,\ K^* = \{A^T v|v \in R^p\}$$

$$\nabla f_0(\bar{x}) + A^T v = 0$$

Let $K_1 = \{u|Au \geq 0\}$

$K_2 = \{u|-Au \geq 0\}$

$K = K_1 \cap K_2 = \{u|Au \geq 0, -Au \geq 0\}$

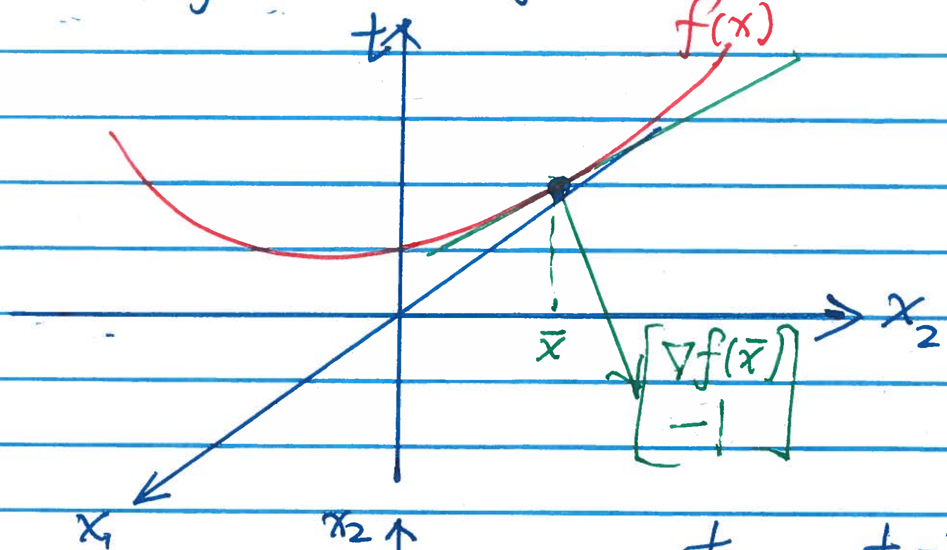
We have

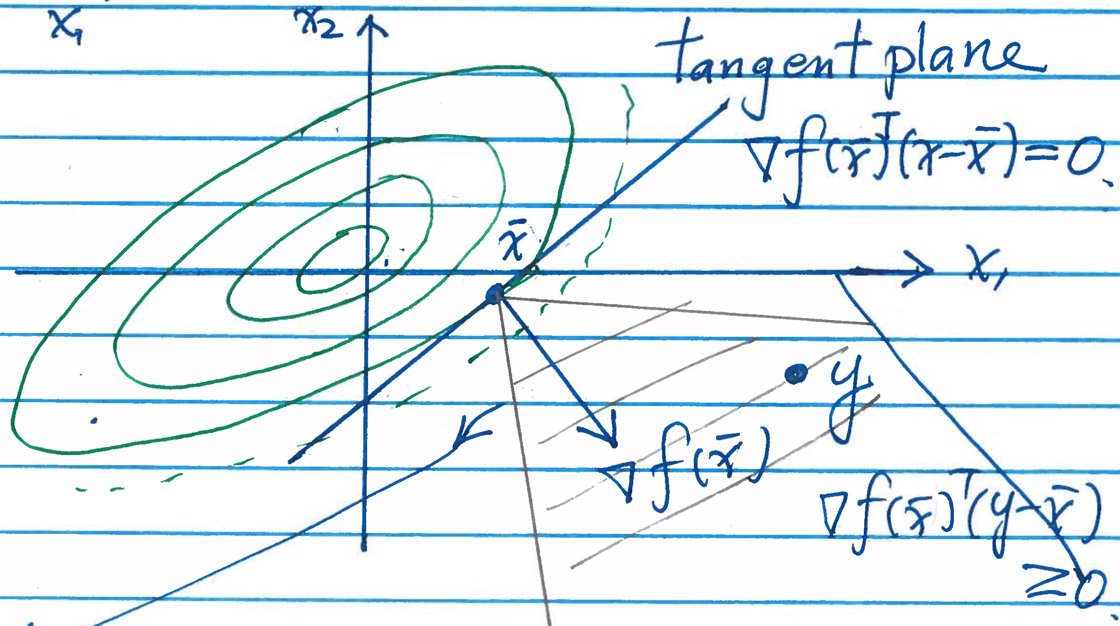
$$K^* = (K_1 \cap K_2)^* = \{A^T v_1 + (-A)^T v_2 | v_1, v_2 \geq 0\}$$

$$= \{A^T v|v \in R^p\}$$

① Hyperplane

$$t(x) = f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x})$$

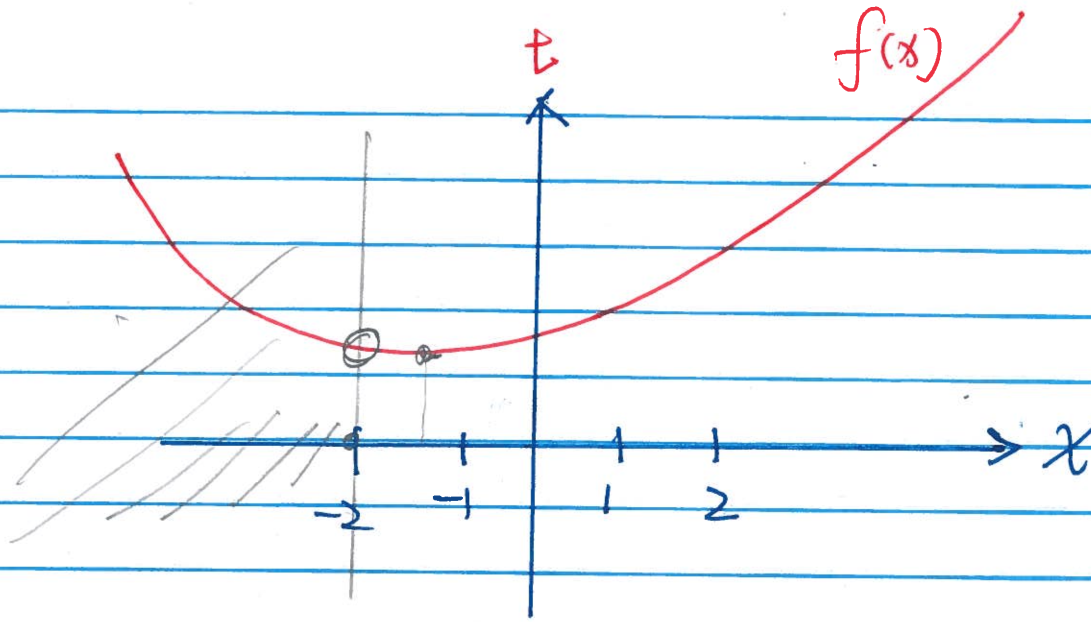




$K = \{ y - \bar{x} \mid y$ in feasible set $\}$

☆ $\bar{x}$ is an opt. solution
because $\nabla f(\bar{x})^T (y - \bar{x}) \geq 0 \;\forall y$ in feasible set

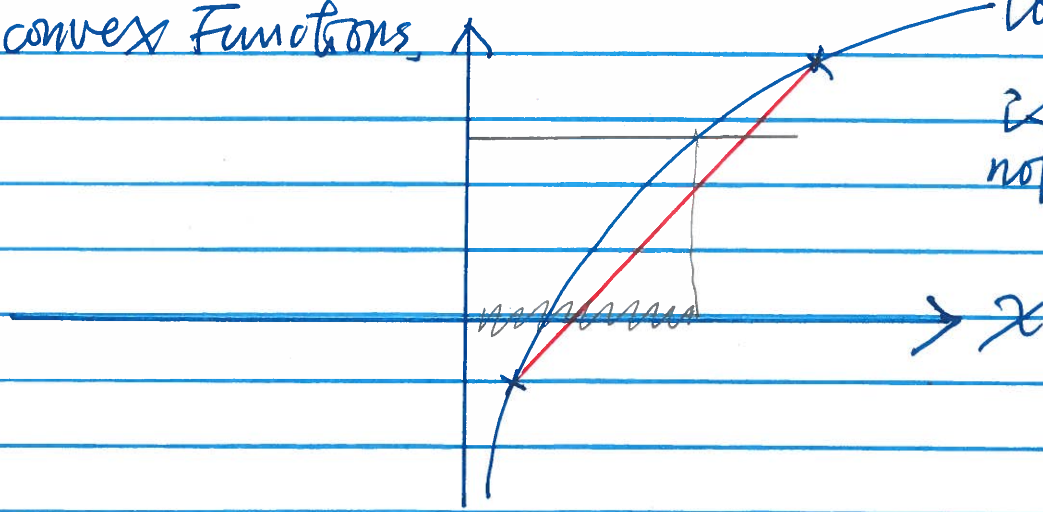
☆ $\bar{x}$ is an opt. solution
if $\nabla f(\bar{x}) \in K^*$, K the cone of feasible set at $\bar{x}$.



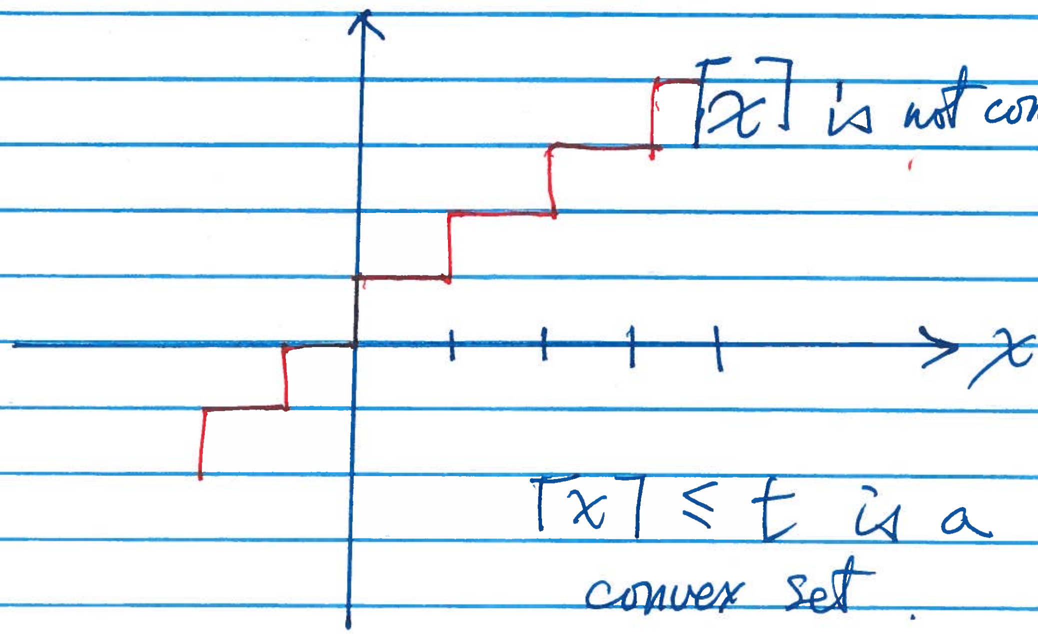
① $x \le -2$ $\nabla f(x) \neq 0 \to \nabla f(x) + A^T v = 0$

② $x \le 2 \to \nabla f(x) = 0$

Quasiconvex Functions



$\log x \leq t \rightarrow$ convex set



$\lceil x \rceil \leq t$ is a convex set.

2.2.3 Opt. with Equality Constraints: Example

$$\min_{x} f(x) = x_1^2 + x_2^2$$

$$s.t.\ [2\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3$$

We can derive $x^* = (x_1^*, x_2^*) = (\frac{6}{5}, \frac{3}{5})$

$$\nabla f(x^*) = \begin{bmatrix} 2x_1^* \\ 2x_2^* \end{bmatrix} = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix},\ \nabla f(x^*) + A^T v = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \times \left(-\frac{6}{5}\right) = 0$$

$(u : Au = b)$

New Problem:

$K = \{ u \mid Au = 0 \}$

$$\begin{aligned} \nabla f(x) + A^T v &= 0 \\ Ax &= b \end{aligned} \Rightarrow \begin{aligned} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} v &= 0 \\ [2\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 3 \end{aligned}$$

$K^* = \{ A^T v \mid v \in R \}$

$\downarrow$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

2.3 Quasiconvex Functions

$f: R^n \to R$ is called quasiconvex (unimodal)
sublevel set $S_t = \{x \mid x \in dom\ f,\ f(x) \le t\}$
if its domain and all sublevel sets $S_t, \forall t \in R$ are convex,
$f: R^n \to R$ is called quasiconcave if $-f$ is quasiconvex.

$f(x)$ quasiconvex and quasiconcave → quasilinear

Ex: $\log x, x \in R_{++}$

2.3 Quasiconvex Functions

Ex: Ceiling function

$Ceil(x) = \inf\{z \in Z | z > x\}$: quasilinear

Ex: $f(x_1, x_2) = x_1 x_2 = \frac{1}{2}[x_1 \quad x_2]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

is quasiconcave in R_+^2, $S_t = \{x \in R_+^2 | x_1 x_2 \geq t\}$

Ex: $f(x) = \frac{a^T x + b}{c^T x + d}$ for $c^T x + d > 0$

$S_t = \{x | c^T x + d > 0, a^T x + b \leq t(c^T x + d)\}$

open halfspace   closed halfspace

→ S_t is convex (t is given here)

→ $f(x)$ is $\left.\begin{matrix}\text{quasiconvex} \\ \text{quasiconcave}\end{matrix}\right\}$ → quasilinear

2.3 Quasiconvex Optimization

min $f_o(x)$ ($f_o(x)$ is quasiconvex, f_i's are convex.)

$s.t. \;\; f_i(x) \leq 0, i = 1, \dots, m$

$Ax = b$

Remark: A locally opt. solution $(x, f_0(x))$ may not be globally opt.

Algorithm: Bisection method for quasiconvex optimization.

Given $l \leq p^* \leq u, \epsilon > 0$

Repeat 1. $t = (l + u)/2$

Find a convex function

2. Find a feasible solution x:

$s.t. \Phi_t(x) \leq 0$ ($f_0(x) \leq t \Leftrightarrow \Phi_t(x) \leq 0$)

$f_i(x) \leq 0$

$Ax = b$

3. If solution is feasible, $u = t, else\ l = t$

Until $u - l \leq \epsilon$

Ex: $f(x) = \frac{p(x)}{q(x)} \leq t \rightarrow p(x) - tq(x) \leq 0$ (p is convex & q is concave)

3. Linear Programming: Format

General Form :

$$\min c^T x$$
$$s.t. \;\; Gx \le h, \;\; G \in R^{m*n}, A \in R^{p*n}$$
$$Ax = b$$

Standard Form :

$$\min c^T x$$
$$s.t. \;\; Ax = b$$
$$x \ge 0$$

Remark: Figure out three possible situations

1. No feasible solutions
2. Unbounded solutions
3. Bounded solutions

3. Linear Programming: Cases

$\min c^T x$

$s.t. \;\; Ax = b$

(1) No feasible solutions: $b \notin R(A)$ (b is not in the range of A)

e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

(2) Unbounded solutions: $b \in R(A)$ but $c \notin R(A^T)$

e.g. $\min \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\nabla f_0(x) + A^T \nu = 0, \nu \in R^m.$

$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2$ (The solution → $-\infty$)

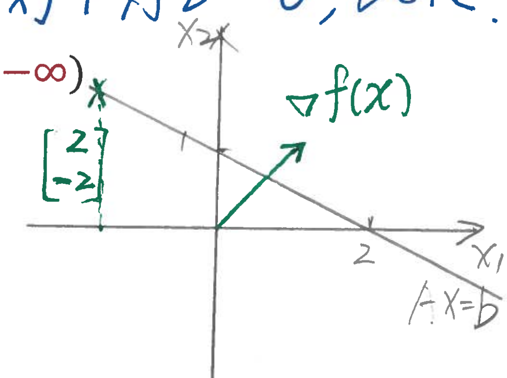
(3) Bounded solutions: b$\in R(A), c \in R(A^T)$

$\begin{bmatrix} -100 \\ 51 \end{bmatrix}$

e.g. $\min \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Thus $x^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, f(x^*) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2$



3. Linear Fractional Programming

P1: $\min f_o(x) = \frac{c^T x + d}{e^T x + f}$, $dom\ f_o = \{x | e^T x + f > 0\}$

$$\begin{aligned} s.t.\ & Gx \leq h \\ & Ax = b \end{aligned}$$

P1⇒P2: Let $y = \frac{x}{e^T x + f}$, $z = \frac{1}{e^T x + f}$

P2: $\min c^T y + dz$

$$\begin{aligned} s.t.\ & Gy - hz \leq 0 \\ & Ay - bz = 0 \\ & e^T y + fz = 1 \\ & z \geq 0 \end{aligned}$$

4. Quadratic Opt. Problems (QP)

QP : $\min \frac{1}{2} x^T P x + q^T x + r$

$$\begin{aligned} s.t.\ & Gx \preccurlyeq h \\ & Ax = b \end{aligned}$$

$P \in S_+^n,\ G \in R^{m \times n},\ A \in R^{p \times n}$

QCQP : (Quadratically Constrained Quadratic Program)

$$\min \frac{1}{2} x^T P_o x + q_o^T x + r_o$$

$$\begin{aligned} s.t.\ & \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, i = 1, \dots, m \\ & Ax = b \end{aligned}$$

$P_i \in S_+^n, i = 0,1, \dots, m$