CSE203B Convex Optimization: Chapter 4: Problem Statement

Obj: $f_0(x)$

s.t. $\underline{\text{Set}}$

defined by

functions = Domain

Feasible Set

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Convex Optimization Formulation

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1. Introduction

Formulation: One of the most critical processes to conduct a project.

$$\begin{aligned}
&\min\ f_0(x)\\
&s.t.\ f_i(x) \leq 0\ \ i = 1, \dots, m\\
&\qquad h_i(x) = 0\ i = 1, \dots, p\ (Ax = b \text{ Affine set})\\
&x \in R^n\\
&D_{f_0}\ f_0: R^n \to R\\
&D_{f_i}\ f_i: R^n \to R\\
&D_{h_i}\ h_i: R^n \to R\\
&f_0, f_i, \dots, f_m\ are\ convex
\end{aligned}$$

$\leq$

$\geq$

$D = \cap_{i=0,m} D_f \cap_{i=0,p} D_{h_i}$ Domain of functions, but not the feasible set.

Feasible Set: The set which satisfies the constraints (is convex for convex problems).

1.1 Introduction: Eliminating Equality Constraints

$$\begin{aligned}
&\min\ f_0(x)\\
&s.t.\ f_i(x) \leq 0\ \ i = 1, \dots, m\\
&\qquad Ax = b
\end{aligned}$$

a. Convert $\{x|Ax = b\}\ to\ \{Fz + x_0|z \in R^k\}$

b. We have a equivalent problem

$$\begin{aligned}
&\min\ f_0(Fz + x_0)\\
&s.t.\ f_i(Fz + x_0) \leq 0
\end{aligned}$$

Let $Ax_0 = b$.
Fz in the nullspace of A

Remark: Matrix F contains columns of null space basis

1.2 Introduction: Slack Variables

$$\min \ f_0(x)$$
$$s.t. \ f_i(x) \leq 0, i = 1, \dots, m$$
$$Ax = b$$

Add slack variables to convert to an equivalent problem

a. Convert the objective function with variable t

$$\min \ t$$
$$s.t. \ f_0(x) - t \leq 0$$
$$f_i(x) \leq 0, i = 1, \dots, m$$
$$A^T x = b$$

b. Convert the inequality with variables s_i

$$\min \ f_0(x)$$
$$s.t. \ f_i(x) + s_i = 0$$
$$A^T x = b$$
$$s_i \in R_+, i = 1, \dots, m$$

1.3 Introduction: Absolute values and Softmax

a. Absolute values

$$|f_i(x)| \leq b$$
$$\Rightarrow f_i(x) \leq b \ and$$
$$-f_i(x) \leq b$$

b. Maximum values

$$\max\{f_1, f_2, \dots, f_m\}$$

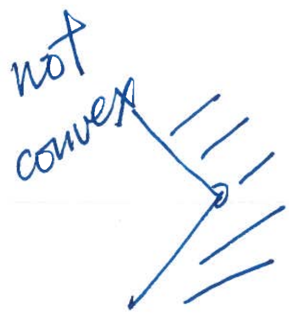
Softmax: $\frac{1}{\alpha}\log\left(e^{\alpha f_1} + e^{\alpha f_1} + \cdots + e^{\alpha f_m}\right)$ $\quad \alpha > 0$

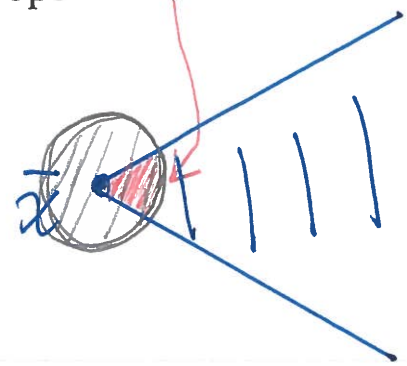
Example: $\max\{1, 5, 10, 2, 3\} \Rightarrow$ Softmax

$$\frac{1}{\alpha}\log(e^{\alpha} + e^{5\alpha} + e^{10\alpha} + e^{2\alpha} + e^{3\alpha}) \approx 10$$

2.1 Optimality Conditions: Local vs. Global Optima

Definition: Local Optima
Given a convex optimization problem and a point $\bar{x} \in R^n$
If there exists a $r > 0$
$s.t.\ \ f_0(z) \geq f_0(\bar{x})$ for all $z \in$ Feasible Set, and $\|z - \bar{x}\|_2 \leq r$
Then $\bar{x}$ is a local optimum.

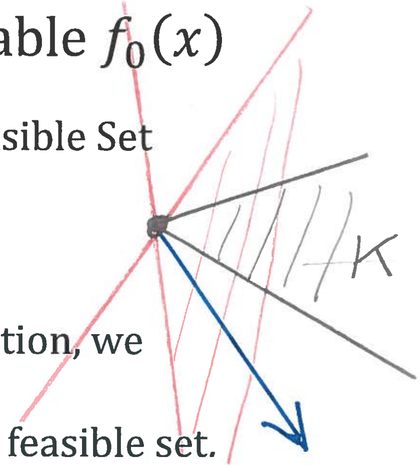




2.2 Optimality Conditions

Theorem: Given a convex opt. problem
If $\bar{x}$ is a local optimum, then $\bar{x}$ is a global optimum
Proof: By contradiction
Suppose that $\exists y \in \mathit{Feasible\ Set}$
$s.t.\ \ f_0(\bar{x}) > f_0(y)$
We have $f_0(\bar{x}) > (1-\theta)f_0(\bar{x}) + \theta f_0(\bar{y})$ (*by assumption*)
$> f_0((1-\theta)\bar{x} + \theta\bar{y})$ (f_0 *is convex*)
And $(1-\theta)\bar{x} + \theta\bar{y}$ is feasible (Feasible set is convex)
The inequality contradicts to the assumption of local optima.

2.2 Optimality Criterion for Differentiable $f_0(x)$



Theorem: If $\nabla f_0(\bar{x})^T(y-\bar{x}) \geq 0$, for a given $\bar{x}$ ∈Feasible Set and for all $y \in$ Feasible Set, then $\bar{x}$ is optimal.
(i.e. $K = \{y - \bar{x} | y \in feasible\ set\}, \nabla f_0(\bar{x}) \in K^*$)

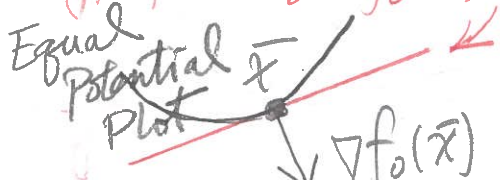
Proof: From the first order condition of convex function, we have $f_0(y) \geq f_0(\bar{x}) + \nabla f_0(\bar{x})^T(y-\bar{x})$.
Given the condition that $\nabla f_0^T(\bar{x})(y-\bar{x}) \geq 0$, $\forall y$ in feasible set.
We have $f_0(y) \geq f_0(\bar{x})$, $\forall y$ in feasible set, which implies that $\bar{x}$ is optimal.

$K = \{\sum_i u_i \theta_i \mid \theta \geq 0\}$

Remark: $\nabla f_0^T(x)(y-x) = 0$ is a supporting hyperplane to feasible set at x.

$f_0(x) \geq f_0(\bar{x}) + \nabla f_0(\bar{x})^T(x-\bar{x})$

Supporting Hyperplane $t = f_0(\bar{x}) + \nabla f_0(\bar{x})^T(x-\bar{x})$



2.2.1 Optimality Criterion without Constraints

Theorem: For problem min $f_0(x), x \in R^n$, where f_0 is convex, the optimal condition is $\nabla f_0(x) = 0$.

Proof: ($\nabla f_0(x) = 0 \Rightarrow$ Optimality)
Since $f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x), \forall x, y \in R^n$ (first order condition of convex function)
We have $f_0(y) \geq f_0(x)$.
Therefore, x is an optimal solution.
($\nabla f_0(x) = 0 \Leftarrow$ Optimality) By contradiction

2.2.2 Opt. with Inequality Constraints

Problem: Min $f_0(x)$

$$s.t.\ Ax \le b, A \in R^{m\times n}$$

Suppose that $A\bar{x} = b$ (one particular case).
Let $x = \bar{x} + u$.

We can write $\begin{cases} \min f_0(\bar{x}+u) \\ \quad Au \le 0 \end{cases}$

Opt. condition: $\nabla f_0(x)^T u \ge 0,\ \forall\{u|Au \le 0\} \equiv K$
In other words,

$$\nabla f_0(\bar{x}) \in K^*\ of\ K = \{u|Au \le 0\}\ and\ K^* = \{-A^T v|v \ge 0\}$$

i.e. $\nabla f_0(\bar{x}) = -A^T v, \exists v \in R^m_+$

$$\nabla f_0(\bar{x}) + A^T v = 0,\ v \ge 0.$$

2.2.3 Opt. with Equality Constraints

$$\begin{cases} \min f_0(x) \\ s.t.\ Ax = b \end{cases}$$

Let $x = \bar{x} + u$ and $A\bar{x} = b$,

we have $\begin{cases} \min f_0(\bar{x}+u) \\ \quad Au = 0 \end{cases}$, $K = \{u|Au = 0\}$

$$\nabla f_0(\bar{x}) \in K^*,\ K^* = \{A^T v|v \in R^p\}$$

$$\nabla f_0(\bar{x}) + A^T v = 0$$

Let $K_1 = \{u|Au \ge 0\}$

$K_2 = \{u|-Au \ge 0\}$

$K_1 \cap K_2 = \{u|Au \ge 0, -Au \ge 0\}$

We have

$$(K_1 \cap K_2)^* = \{A^T v_1 + (-A)^T v_2|v_1, v_2 \ge 0\}$$
$$= \{A^T v|v \in R^p\}$$