Recap

1. Taylor's Exp.

$$f(x) = \underline{f(\bar{x}) + \nabla f(\bar{x})^T(x-\bar{x})} + (x-\bar{x})^T \nabla^2 f(x)(x-\bar{x})$$

2. 1'st Order Cond.

$$f(x) \geq \underline{f(\bar{x}) + \nabla f(\bar{x})^T(x-\bar{x})}$$

3. Supporting Hyperplane.

$$\nabla f(\bar{x})^T \bar{x} - f(\bar{x}) \geq \nabla f(\bar{x})^T x - f(x)$$

$$\cancel{\begin{bmatrix} \nabla f(\bar{x}) \\ -1 \end{bmatrix}^T \begin{bmatrix} \bar{x} \\ f(x \end{bmatrix}} \begin{bmatrix} \nabla f(\bar{x}) \\ -1 \end{bmatrix}^T \begin{bmatrix} \bar{x} \\ f(\bar{x}) \end{bmatrix} \geq \begin{bmatrix} \nabla f(\bar{x}) \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ f(x) \end{bmatrix}$$

$$\bar{b} = a^T \begin{bmatrix} x \\ t \end{bmatrix}$$

4. Conjugate Function.

$$\underline{f^*(y) = \max_x y^T x - f(x) = \max_x \begin{bmatrix} y \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ f(x) \end{bmatrix}}$$

Let $y = \nabla f(\bar{x})$, we have

$$f^*(y) = \begin{bmatrix} y \\ -1 \end{bmatrix}^T \begin{bmatrix} \bar{x} \\ f(\bar{x}) \end{bmatrix} \quad (\bar{b})$$

$$t = \nabla f(\bar{x})^T x - \bar{b}$$

$$= y^T x - f^*(y)$$

Chat: $-f^*(y) = -[\min(f(x) - y^T x)]$ $\left(\begin{array}{l} \min f(x) \\ \text{s.t. } x \leq 0. \end{array}\right)$

4. Conjugate Functions

Suppose we have a pair $\bar{x}, \bar{y}$, such that $f^*(\bar{y}) = \bar{y}^T\bar{x} - f(\bar{x})$, we can show that $\bar{y} = \nabla_x f(\bar{x})$ (exercise 3.40)

And the supporting hyperplane : $\bar{y}^T x - h = f^*(\bar{y})$

$$[\bar{y}^T \quad -1]\begin{bmatrix} x \\ h \end{bmatrix} = f^*(\bar{y})$$

Ex. $f(x) = x^2 - 2x, \quad x \in R$

$$f^*(y) = \sup_x yx - x^2 + 2x, y \in R$$

$$g(x,y) = yx - x^2 + 2x$$

$$\frac{d\, g(x,y)}{dx} = y - 2x + 2 \Rightarrow x^* = \frac{y}{2} + 1.$$

$$f^*(y) = \frac{y^2}{4} + y + 1.$$

hyperplane $xy - f^*(y) = xy - \left[\frac{y^2}{4} + y + 1\right]$

4. Conjugate Functions

One way to view conjugate function

$$f^*(y) = \sup_{x \in dom\, f} y^T x - f(x)$$

x : negative slack

y : shadow price (loss) to accommodate the slack

$f^*(y)$: balance between price slack product ($y^T x$) and objective function $f(x)$.

Remark: When $f^*(y)$ is unbounded, the shadow price y is not reasonable.

4. Conjugate Functions: Examples (single variable)

Ex: $f(x) = ax + b, \ x \in R$

$f^*(y) = \sup_x (yx - ax - b) = \max_{x \in R} (y-a)x - b$

(1) If $y \neq a, f^*(y) = \infty$

(2) If $y = a, f^*(y) = -b \rightarrow dom\ f^* = a, f^*(y) = -b$

$f(x) = 3x + 2.$

$f^*(y=5) = \max_x y^T x - f(x)$

$= \max 5x - (3x+2)$

$\cancel{= \max 8x + 2}$

$= \max_{x \in R} 2x - 2$

$\rightarrow \infty \ |_{x \to \infty}.$

4. Conjugate Functions: Examples (single variable)

Ex: $f(x) = -logx, \ \ x \in \mathrm{R}_{++}$

$f^*(y) = \sup_{x \in R_{++}} yx + logx$

Let $y = 5.$

$f^*(y=5) = \max_x 5x + \log x.$

(1) If $y \geq 0, \ f^*(y) = \infty$

(2) If $y < 0, \ f^*(y) = \max_{x \in R_{++}} xy + logx$

Let $g(x) = xy + logx, \ g'(x) = y + \frac{1}{x}$

If $g'(x) = 0, x = -\frac{1}{y}$

Thus, $f^*(y) = -1 + \log\left(-\frac{1}{y}\right) = -1 - \log(-y)$

$\rightarrow dom\ f^* = -R_{++}, \ f^*(y) = -1 - \log(-y)$

$t = xy - f^*(y) = xy - [-1 - \log(-y)]$

4. Conjugate Functions

Ex: $f(x) = e^x,\ x \in R$

$f^*(y) = \sup_x xy - e^x$

(1) $y < 0 : f^*(y) = \infty$

(2) $y > 0$: Let $g(x) = xy - e^x \rightarrow g'(x) = y - e^x$

If $g'(x) = 0$, then $x^* = logy$

Thus $f^*(y) = ylogy - y$

(3) $y = 0 : f^*(y) = 0 \rightarrow dom\ f^* = R_+,\ f^*(y) = ylogy - y$

Therefore, we have

$f^*(y) = ylogy - y$, where $y \geq 0$.

hyperplane $xy - f^*(y)$

4. Conjugate Functions

Ex: $f(x) = xlogx,\ x \in R_+,\ f(0) = 0$

$f^*(y) = \sup_x xy - xlogx$

Let $g(x) = xy - xlogx \rightarrow g'(x) = y - logx - 1$

Suppose $g'(x) = 0$, we have $y = 1 + logx$ or $x = e^{y-1}$

Thus $f^*(y) = ye^{y-1} - e^{y-1}(y-1) = e^{y-1}$ where $y \in R$

4. Conjugate Functions

Ex: $f(x) = \frac{1}{2}x^T Qx,\ \ x \in R^n, Q \in S^n_{++}$

$$f^*(y) = \sup_x x^T y - \frac{1}{2}x^T Qx$$

Let $g(x) = x^T y - \frac{1}{2}x^T Qx \rightarrow \nabla g(x) = y - Qx$

If $\nabla g(x) = 0$, we have $x = \underline{Q^{-1}y}$

$$g(x) = (Q^{-1}y)^T y - \frac{1}{2}(Q^{-1}y)^T Q (Q^{-1}y)$$

$$= y^T Q^{-1} y - \frac{1}{2} y^T \underbrace{Q^{-T} Q Q^{-1}}_{} y$$

$$= y^T Q^{-1} y - \frac{1}{2} y^T Q^{-1} y$$

$$= \frac{1}{2} y^T Q^{-1} y$$

Thus, $f^*(y) = \frac{1}{2}y^T Q^{-1} y$

Remark: Suppose that $f^*(\bar{y}) = \bar{y}^T \bar{x} - f(\bar{x})$ and $\nabla^2 f(\bar{x}) \succ 0$

We have $\nabla f^*(\bar{y}) = \bar{x}$ and $\nabla^2 f^*(\bar{y}) = \left(\nabla^2 f(\bar{x})\right)^{-1}$ (exercise 3.40)

4. Conjugate Functions

Basic Properties

(1) $f(x) + f^*(y) \geq x^T y$

Fenchel's inequality. Thus, in the above example

$$x^T y \leq \frac{1}{2}x^T Qx + \frac{1}{2}y^T Q^{-1} y,\ \ \forall x, y \in R^n, Q \in S^n_{++}$$

(2) $f^{**} = f$, if f is convex & f is closed (i.e. *epi* f is a closed set)

(3) If f is convex & differentiable, $\mathit{dom}\ f = R^n$

For $\max y^T x - f(x)$, we have $y = \nabla f(x^*)$

Thus, $f^*(y) = x^{*T} \nabla f(x^*) - f(x^*),\ \ y = \nabla f(x^*)$