3. Operations that preserve convexity: Dual norm

Example:

$$f(x) = \max_{\|y\|_2 \le 1} y^T x$$

$$f(x) = \max_{\|y\|_1 \le 1} y^T x$$

$$f(x) = \max_{\|y\|_p \le 1} y^T x$$

$$\|y\|_p = \left(\sum_i |y_i|^p\right)^{1/p}.$$

3. Operations that preserve convexity: max function

Theorem: Pointwise maximum of convex functions is convex
Given $f(x) = \max\{f_1(x), f_2(x)\}$, where f_1 and f_2 are convex and $dom\, f = dom\{f_1\} \cap dom\{f_2\}$ is convex, then $f(x)$ is convex.

Proof: For $0 \le \theta \le 1,\ x, y \in dom\, f$

$f(\theta x + (1-\theta)y)$

$= \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\}$ def of f.

$\le \max\{\theta f_1(x) + (1-\theta)f_1(y), \theta f_2(x) + (1-\theta)f_2(y)\}$ convex f_1, f_2

$\le \theta \max\{f_1(x),\ f_2(x)\} + (1-\theta)\max\{f_1(y), f_2(y)\}$ triangular ineq. of $\| \|_\infty$

$= \theta f(x) + (1-\theta)f(y)$ def of f .

i.e. $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y)$

Thus, function $f(x)$ is convex.

3. Operations that preserve convexity: minimization

Theorem: Partial minimization

If $g(x, y)$ is convex in x and y, and a set C is convex

Then $f(x) = \min_{y \in C} g(x, y)$ is convex.

Proof: Let $y_1 \in \{y | \min_{y \in C} g(x_1, y)\}$ and $y_2 \in \{y | \min_{y \in C}(g(x_2, y)\}$,

we can write

$$
\begin{aligned}
&\theta f(x_1) + (1-\theta) f(x_2) \\
&\quad = \theta g(x_1, y_1) + (1-\theta) g(x_2, y_2) \quad \text{notation of } y_1 \ \& \ y_2 \\
&\quad \geq g(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta) y_2) \ (g \ is \ convex) \\
&\quad \geq \min_{y \in C} g(\theta x_1 + (1-\theta)x_2, y) \ (C \ is \ convex) \\
&\quad = f(\theta x_1 + (1-\theta)x_2) \quad \text{def of } f.
\end{aligned}
$$

i.e. we have $\theta f(x_1) + (1-\theta) f(x_2) \geq f(\theta x_1 + (1-\theta)x_2)$

Therefore, $f(x) = \min_{y \in C} g(x, y)$ is convex.

3. Operations that preserve convexity

Examples for Partial Minimization

Given $f(x, y) = [x^T \quad y^T] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$x \in R^n, y \in R^m, A \in S_+^n, C \in S_+^m, \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in S_+^{n+m}$$

Let $g(x) = \min_y f(x, y) = x^T (A - BC^+ B^T) x$,

C^+: **pseudo inverse** of matrix C. (Drazin inverse, or generalized inverse)

We can claim that function $g(x)$ is convex.

Proof:

(1) $f(x, y)$ is convex

(2) $y \in R^m$ where R^m is a convex non-empty set

(3) Therefore, $g(x)$ is convex, i.e. $A - BC^+ B^T \succcurlyeq 0$

$$\max\left(\overset{a}{\theta f_1(x)} + \overset{c}{(1-\theta) f_1(y)},\ \overset{b}{\theta f_2(x)} + \overset{d}{(1-\theta) f_2(y)}\right)$$

$$\leq \theta \max\left(\underset{a}{f_1(x)}, \underset{b}{f_2(x)}\right) + (1-\theta) \max\left(\underset{c}{f_1(y)}, \underset{d}{f_2(y)}\right)$$

$$\max\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right)$$ ~~= max~~

I, $= \left\| \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} a+c \\ b+d \end{bmatrix} \right\|_\infty$

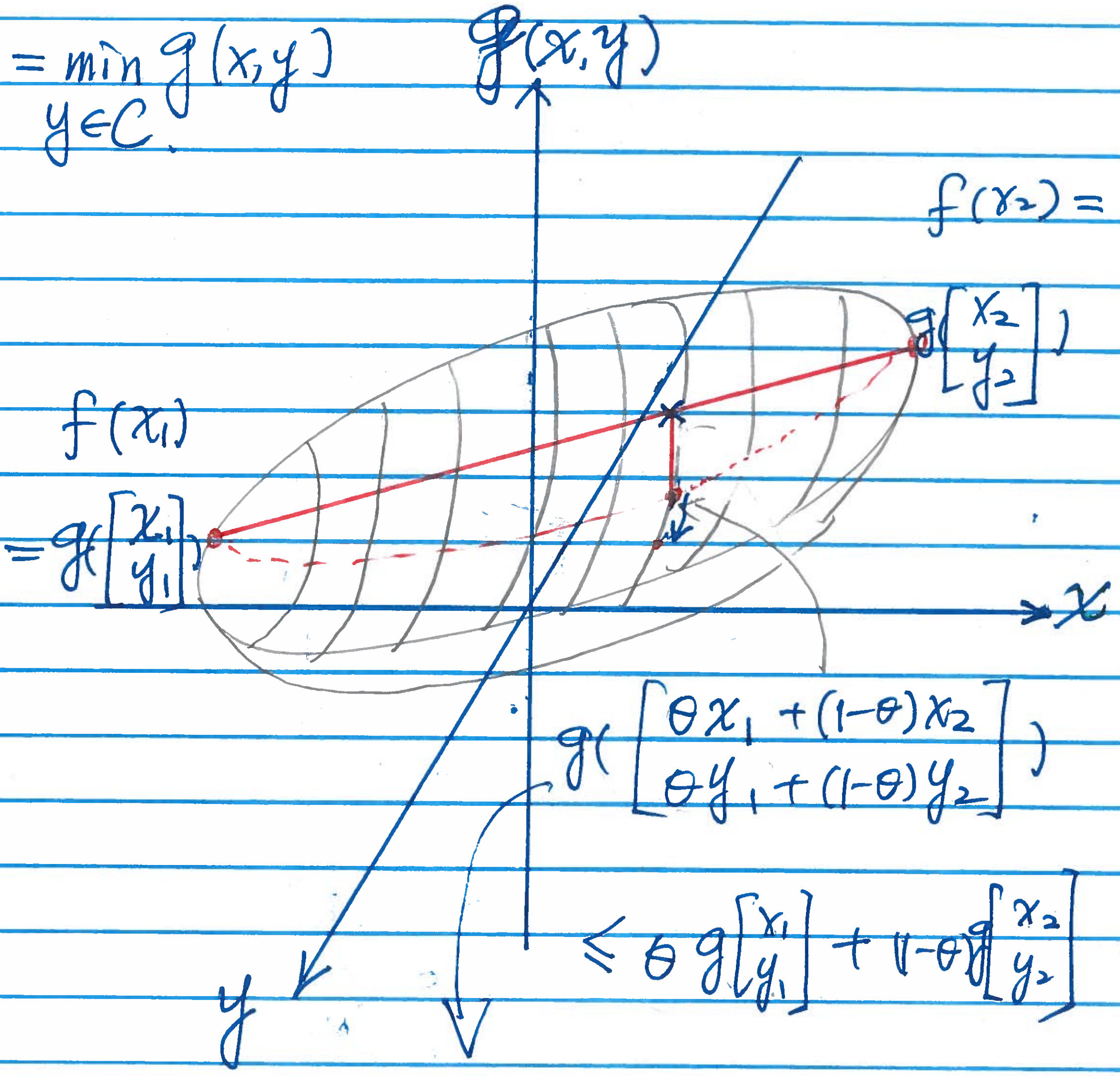
II, $\leq \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_\infty + \left\| \begin{bmatrix} c \\ d \end{bmatrix} \right\|_\infty$

Ex:

I : $\left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\|_\infty = 5.$

II $\left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\|_\infty + \left\| \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\|_\infty = 3 + 4 = 7.$

$$f(x) = \min_{y \in C} g(x, y)$$



$$g\left(\begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta y_1 + (1-\theta) y_2 \end{bmatrix}\right)$$

$$\leq \theta g \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta) g \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad \otimes$$

$$\geq \min_{y \in C} g \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ y \end{bmatrix}$$

$$\theta f(x_1) + (1-\theta) f(x_2) \geq f(\theta x_1 + (1-\theta) x_2)$$

3. Operations that preserve convexity

Composition:

Given $g: R^n \rightarrow R$ and $h: R \rightarrow R,$ we set $f(x) = h(g(x))$

f is convex if g convex, h convex, $\tilde{h}$ nondecreasing

g concave, h convex, $\tilde{h}$ nonincreasing

f is concave if g convex, h concave, $\tilde{h}$ nonincreasing

g concave, h concave, $\tilde{h}$ nondecreasing

Proof : for n=1

$$f''(x) = h''\big(g(x)\big)g'(x)^2 + h'\big(g(x)\big)g''(x)$$

Ex1: exp $g(x)$ is convex if g is convex

Ex2: $1/g(x)$ is convex if g is concave and positive

Note that we set $\tilde{h}(x) = \infty$ if $x \notin dom\ h$, h is convex

$\tilde{h}(x) = -\infty$ if $x \notin dom\ h$, h is concave

3. Operations that preserve convexity

Show that $h\big(g(\theta x + (1-\theta)y)\big) \leq \theta h\big(g(x)\big) + (1-\theta)h(g(y))$

for the case that g, h are convex, and $\tilde{h}$ is nondecreasing

(1) g is convex

$$g(\theta x + (1-\theta)y) \leq \theta g(x) + (1-\theta)g(y)$$

(2) h is nondecreasing: From (1), we have

$$h\big(g(\theta x + (1-\theta)y)\big) \leq h\big(\theta g(x) + (1-\theta)g(y)\big)$$

(3) h is convex

$$h\big(\theta g(x) + (1-\theta)g(y)\big) \leq \theta h\big(g(x)\big) + (1-\theta)h\big(g(y)\big)$$

(4) From (2) & (3)

$$h\big(g(\theta x + (1-\theta)y)\big) \leq \theta h\big(g(x)\big) + (1-\theta)h\big(g(y)\big)$$

4. Conjugate Functions

The setting of conjugate functions starts from the following problem (which may not be convex)

$$min\, f(x)$$

subject to

$$x \leq 0$$

We convert to a function of y

$$\inf_x f(x) - y^T x$$

The conjugate function is

$$f^*(y) = \sup_x y^T x - f(x)$$

In the class, we interchange min and inf; max and sup to simplify the notation.

4. Conjugate Functions

Given $f: R^n \to R$, we have $f^*: R^n \to R$

$$f^*(y) = \sup_{x \in dom\, f} y^T x - f(x); \quad (-f^*(y) = \min_{x \in dom\, f} - y^T x + f(x))$$

Constraint: $y \in R^n$ for which the supremum is finite (bounded)

$f^*(y)$ is called the conjugate of function f

Theorem : $f^*(y)$ is convex (pointwise maximum)

Proof : $f^*(\theta y_1 + (1-\theta)y_2) = \sup_x (\theta y_1 + (1-\theta)y_2)^T x - f(x)$

$$\leq \sup_x \left(\theta y_1^T x - \theta f(x)\right) + \sup_x ((1-\theta) y_2^T x - (1-\theta) f(x))$$

$$= \theta f^*(y_1) + (1-\theta) f^*(y_2)$$

Remark: $f^*(y)$ is convex even if $f(x)$ is not convex

Why we have bounded constraint.

Ex. $f(x) = x_1 + 2x_2$.

$$f^*\left(y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \max_x y^T x - f(x)$$

$$= \max_x \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - (x_1 + 2x_2)$$

$$= \max_x x_1 - (x_1 + 2x_2)$$

$$= \max_x \underline{-2x_2} \rightarrow \infty \text{ if } x_2 \rightarrow -\infty$$

$$f^*(y) = \max_{x} y^T x - f(x)$$

Suppose $f(x)$ is a convex funct.

Let $g(x,y) = y^T x - f(x)$

$$\nabla_x g(x,y) = y - \nabla f(x) = 0$$

$$\Rightarrow y = \nabla f(x).$$

$$f^*(y) = \max_{x} y^T x - f(x) = \max_{x} [y^T, -1] \begin{bmatrix} x \\ f(x) \end{bmatrix}$$

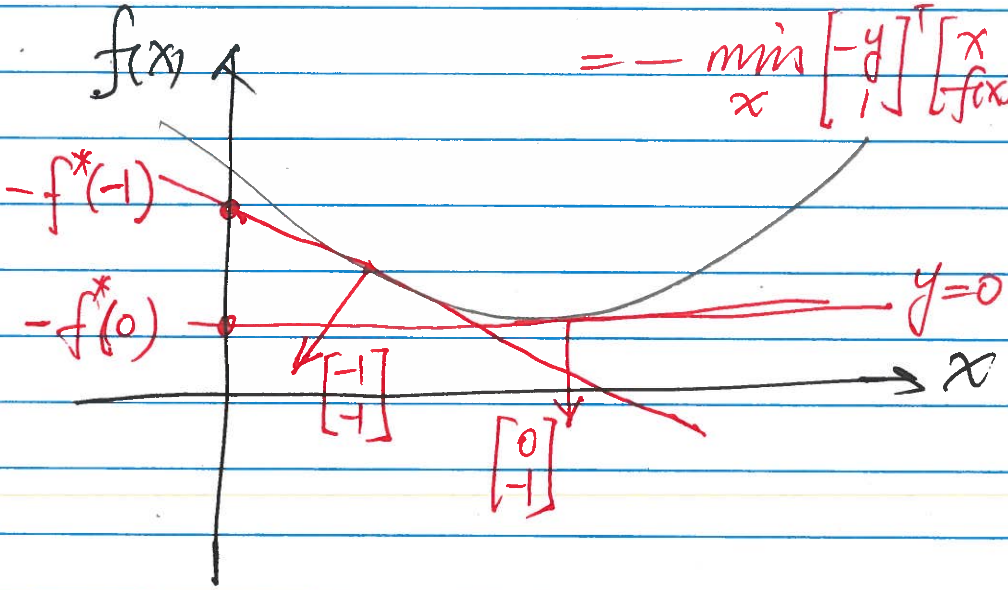
$$= [y^T, -1] \begin{bmatrix} \bar{x} \\ f(\bar{x}) \end{bmatrix}$$

where $\nabla f(\bar{x}) = y$.

Conjugate Function

$$f^*(y) = \max_x y^T x - f(x) = \max_x \begin{bmatrix} y \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ f(x) \end{bmatrix}$$

$$= -\min_x \begin{bmatrix} -y \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ f(x) \end{bmatrix}$$



Supporting hyperplane

$$[y^T \; -1]\left(\begin{bmatrix} x \\ \overset{t}{f(x)} \end{bmatrix} - \begin{bmatrix} x_0 \\ f(x_0) \end{bmatrix}\right) = 0$$

where $y = \nabla f(x_0)$.

or $[y^T \; -1] \begin{bmatrix} x \\ \overset{t}{f(x)} \end{bmatrix} = b$

where $b = [y^T \; -1] \begin{bmatrix} x_0 \\ f(x_0) \end{bmatrix}$

$$= y^T x_0 - f(x_0)$$

$$= -(-y^T x_0 + f(x_0))$$