## 3. Separating Hyperplane

$\{x|a^T x = b\}$ (Classification, Optimization, Duality)

Theorem : Given two convex sets $C \cap D = \emptyset$ in $R^n$

$$\exists a \in R^n, b \in R, \quad s.t.\ a^T x \le b, \forall x \in C$$
$$a^T x \ge b, \forall x \in D$$

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Actually, $a = d - c, b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}$

i.e. $f(x) = a^T x - b = (d-c)^T (x - \frac{d+c}{2})$

For $dist(C, D) = \inf\{\|u - v\|_2 \,|\, u \in C, v \in D\}$



## 3. Separating Hyperplane

Assumption: $dis(c,d) \le dis(u,v)$ $\forall u \in C$, $v \in D$.

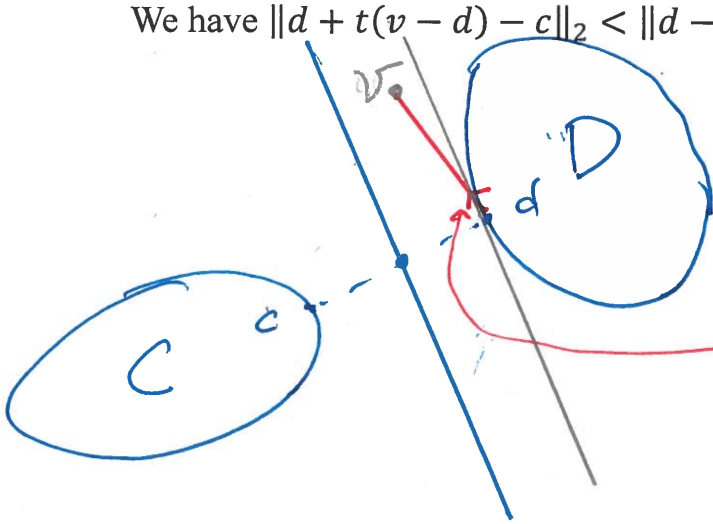
Proof : $\forall v \in D, a^T v \ge a^T d$ should be true

By contradiction, suppose that $a^T v < a^T d$

Then we can show that $d + t(v - d)$ is close to $c$ for $t > 0$

Because $\frac{d}{dt}\|d + t(v-d) - c\|_2^2 = 2(\underbrace{d - c}_{a})^T(v - d) < 0$

We have $\|d + t(v - d) - c\|_2 < \|d - c\|_2$ for tiny $t > 0$




$u \in D$

1. Convexity

$d + t(u-d) \in D \ \forall 1 \ge t \ge 0$

2. Calculus. suppose that $a^T v < a^T d$

$\Rightarrow dis(d + t(u-d), c) < dis(c, d)$

## 3. Supporting Hyperplane

Given set $C \in R^n$, and a point $x_0$ on the boundary of $C$, the hyperplan $\{x|a^T x = a^T x_0\}$ is called supporting hyperplane of $C$ if $a^T x \le a^T x_0, \forall x \in C$.

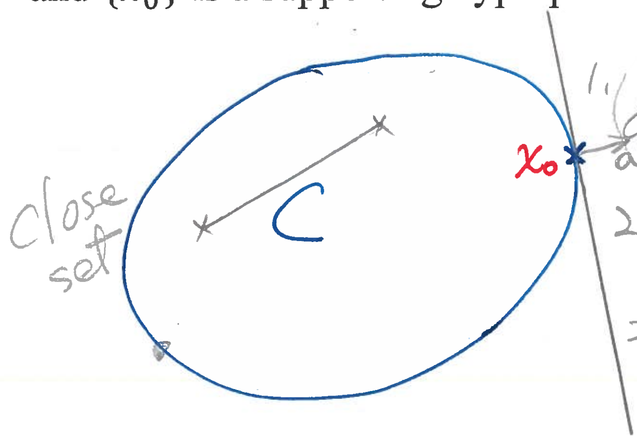
Supporting Hyperplane Theorem: For any nonempty convex set $C$, and a point $x_0$ on the boundary of $C$,

There exists a support hyperplane to $C$ at $x_0$.

$\nabla f_0(x)$

Proof: A separating hyperplane that separates interior $C$ and $\{x_0\}$ is a supporting hyperplane.

Open set




1. disjoint $\tilde{C}$ and $\{x_0\}$.
2. $\tilde{C}$, $\{x_0\}$ are convex sets
3. Separating hyperplane



## 4. Dual Cones

Given Cone $K$ (i.e. $K = \{\sum_{i=1}^{k} \theta_i u_i \,|\theta_i > 0, u_i \in R^n, \forall i\}$)

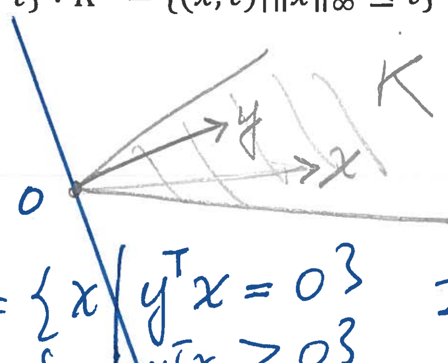
$K^* = \{y|x^T y \ge 0, \forall x \in K\}$

Ex: 1. $K = R_+^n : K^* = R_+^n$

2. $K = S_+^n : K^* = S_+^n$

3. $K = \{(x,t)|\|x\|_2 \le t\} : K^* = \{(x,t)|\|x\|_2 \le t\}$

4. $K = \{(x,t)|\|x\|_1 \le t\} : K^* = \{(x,t)|\|x\|_\infty \le t\}$

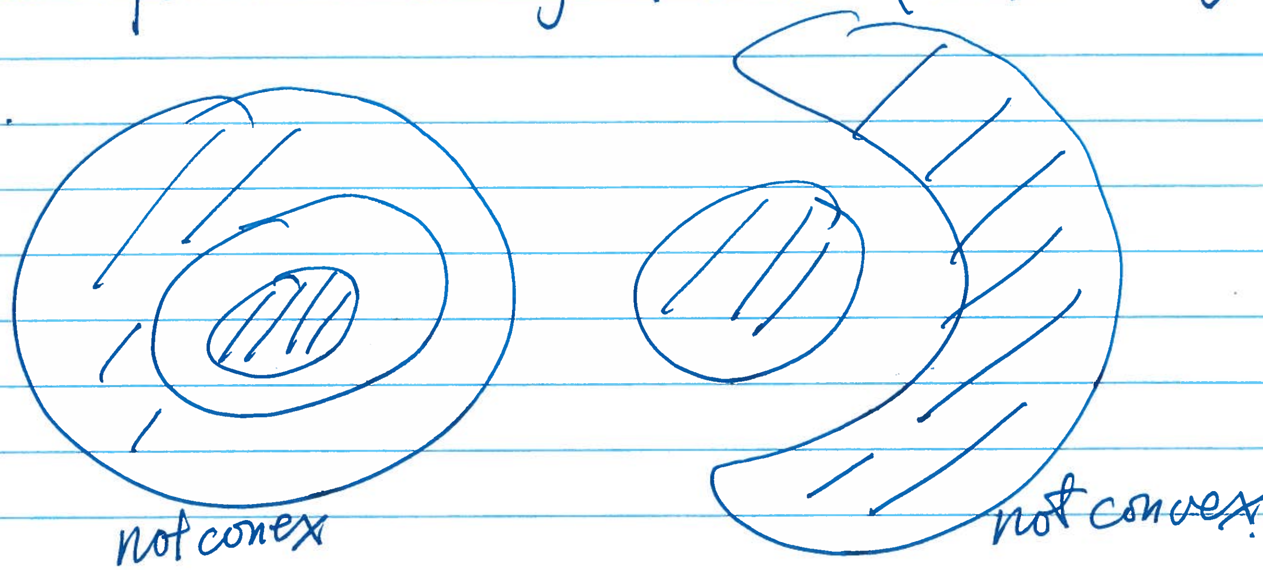



$H = \{x | y^T x = 0\}$
$S_1 = \{x | y^T x > 0\}$
$S_2 = \{x | y^T x < 0\}$

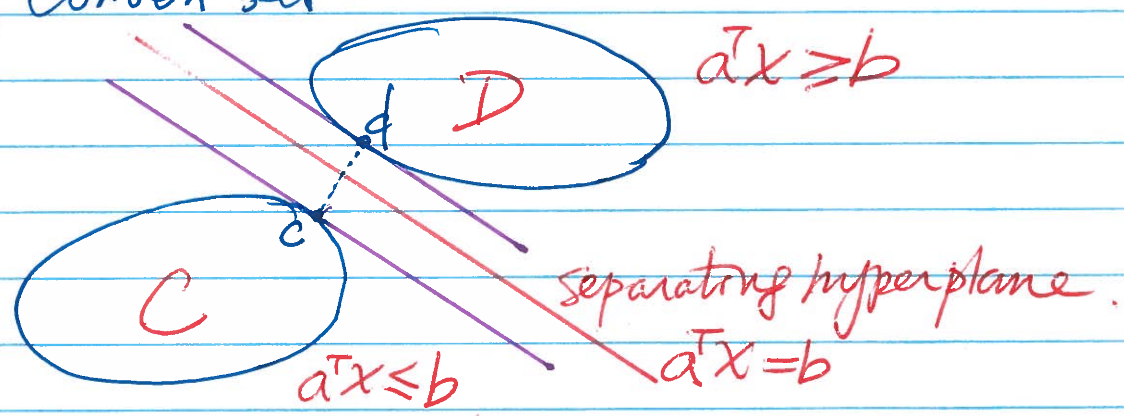
If $K \subset S_1$
Then $y \in K^*$

In general, we may not find a hyperplane that separate two disjoint sets (not convex)

Ex.




Ex. Convex set.




1. Assumption: We can find $c \in C$, $d \in D$ s.t. $\|c-d\|_2 = \inf\{\|u-v\|_2 \mid u \in C, v \in D\}$
2. Def of convexity: $\forall v \in D$, $d + t(v-d) \in D$ $(t \geq 0, t \leq 1)$.
3. Def of separating hyperplane $a^T d \leq a^T v$ $\forall v \in D$.
4. Calculus

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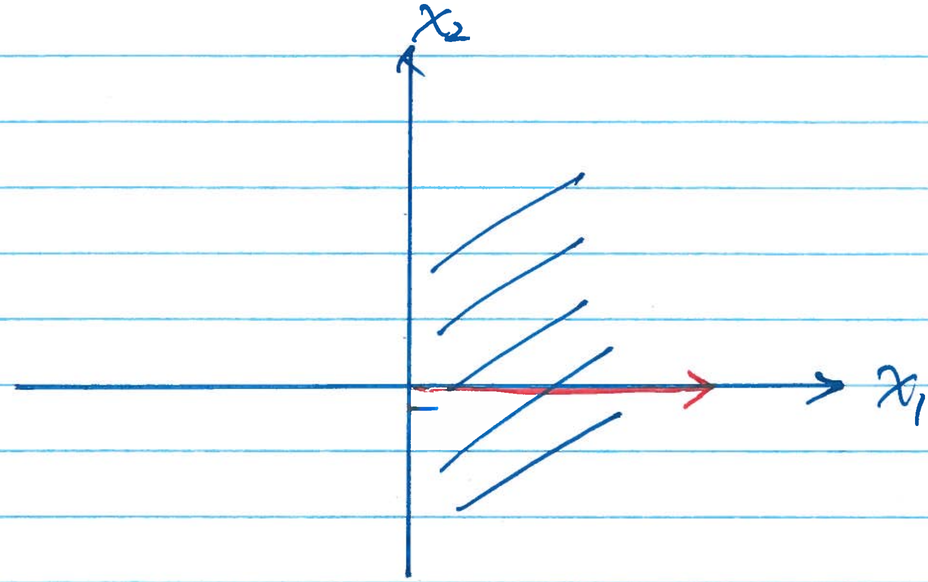
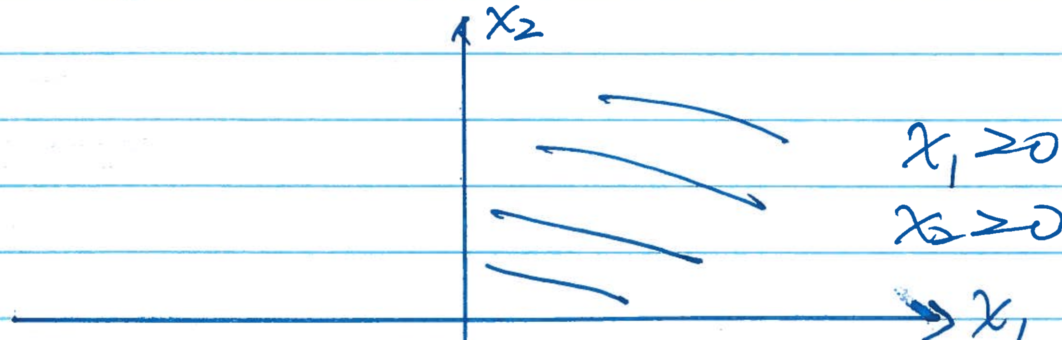



$K = \{ x \mid x_1 \geq 0,\ x \in R^2 \} = \{ x \mid [1\ 0][x] \geq 0, x \in R^2 \}$

$K^* = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \mid \theta \geq 0 \}$




$K = \{ x \mid x_1 \geq 0, x_2 \geq 0 \} = \{ x \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \geq 0, x \in R^2 \}$

$K^* = \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \mid \theta_1, \theta_2 \geq 0 \}$

Cone $K$ $\{\sum \theta_i u_i \mid \theta_i > 0, u_i \in R^n\}$

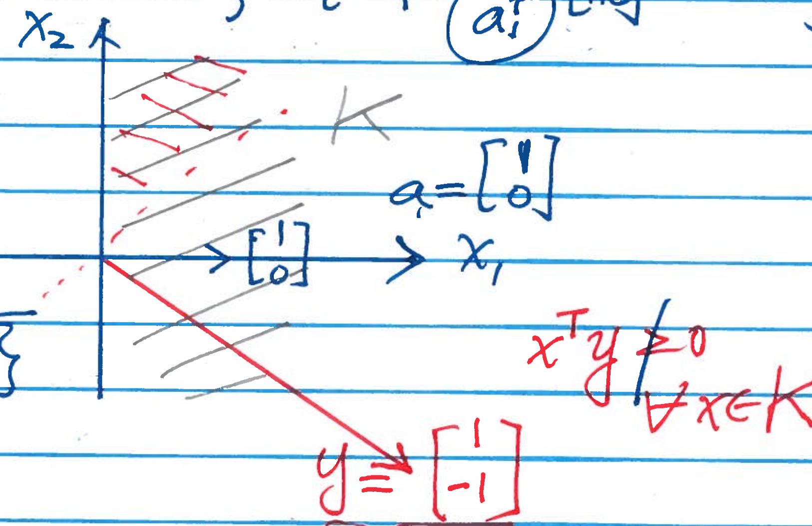
Ex 1. $K^* = \{y \mid x^T y \geq 0 \ \forall x \in K\}$

Ex: $K = \{x \mid x_1 \geq 0, x \in R^2\} = \{x \mid \underbrace{[1 \ 0]}_{a_1^T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0, x \in R^2\}$

$K^* = ?$ $K_1$

$K_2 = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta \mid \theta \geq 0\}$

$K_3 = R^2$




Thus $K^* = K_2 = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta \mid \theta \geq 0\}$

For $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

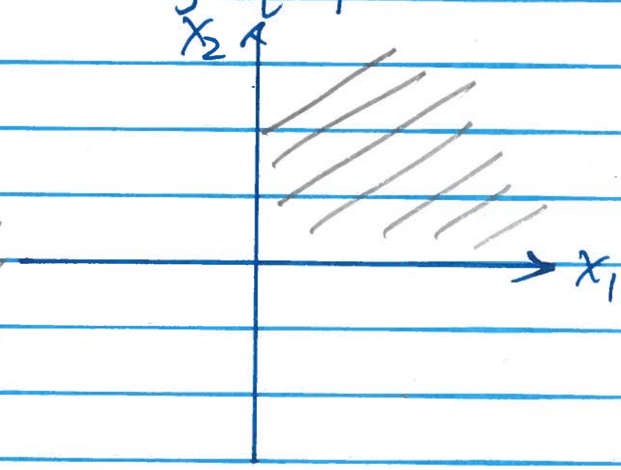
$x^T y \geq 0 \ \forall x \in K$

Ex 2. $K = \{x \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \geq 0, x \in R^2\} = \{x \mid x_1 \geq 0, x_2 \geq 0\}$.

$K^* = ?$

$K$

$K^* = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \mid \theta_1, \theta_2 \geq 0\}$




Thus, $K^* = K$.

# 4. Dual Cones

Show that cone $K = \{(x,t) | \|x\|_1 \leq t\}$ has its dual

$$K^* = \{(x,t) | \|x\|_\infty \leq t\}$$

Proof :

Statement $x^T u + tv \geq 0, \forall \|x\|_1 \leq t \leftrightarrow \|u\|_\infty \leq v$

L=>R By contradiction, suppose that $\|u\|_\infty > v$

We can find $\exists x \ s.t \ \|x\|_1 \leq 1, x^T u > v$

Setting t=1, then we have $u^T(-x) + v < 0.$

R=>L Given $\|x\|_1 \leq t, \|u\|_\infty \leq v$

$u^T \|-x/t\|_1 \leq \|u\|_\infty \leq v$

Thus, $u^T(-x) \leq vt$



# 4. Dual Cones

Definition: $x \leq_K y$ if $y - x \in K$

Theorem: $x \leq_K y$ iff $\lambda^T x \leq \lambda^T y, \forall \lambda \in K^*$

Examples

# 4. Dual Cones

The polyhedral cone $V = \{x|Ax \geq 0\}$ has its dual cone

$V^* = \{A^T v | v \geq 0\}$

Proof : By definition

$$V^* = \{y|x^T y \geq 0, \forall x \in V\}$$

Thus $V^* = \{y|x^T y \geq 0, \forall Ax \geq 0\}$

Let $y = A^T v$, we have $x^T y = x^T A^T v > 0$ if $v \geq 0$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ i.e. $x_1 + 2x_2 \geq 0, x_1 - x_2 \geq 0$

$A^T = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ i.e. $\{\theta_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \theta_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} | \theta_1, \theta_2 \geq 0\}$



# 4. Dual Cones

Remark: $\{x_0 + \Delta x | \Delta x \in K\}$

(1) $K$ cone can be translated to $x_0$

(2) If $a \in K^*$, then $a^T x \geq 0, \forall x \in K$, i.e. $-ax$ is a supporting hyperplane of cone $K$

(3) Suppose that the current feasible search region is at corner $x_0$ and $\{x_0 + \Delta x | \Delta x \in K, ||\Delta x|| < r\}$ is a local feasible region of a convex set

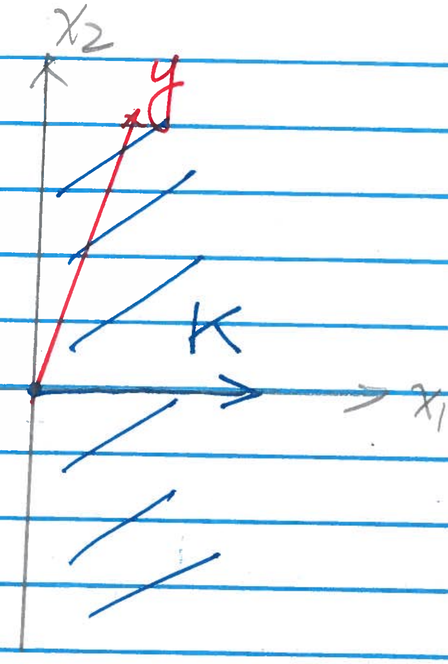
If $\nabla f_0(x_0) \in K^*$, i.e. $\nabla f_0(x_0)^T \Delta x \geq 0, \forall \Delta x \in K$,

Then $x_0$ is an optimal solution

Ex3. $K = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \,\middle|\, \theta \geq 0 \right\}$.

$K^* = ?$ $\quad \{ y \mid y^T x \geq 0, \forall x \in K \}$

$K^* = \{ x \mid [1, 0] x \geq 0 \ \forall x \}$




Given a cone

$K$, $(K^*)^* \stackrel{?}{=} K$

($A \subset B$ but $B \not\subset A$)

Given a cone ~~$K = \{x \mid Ax \leq 0$~~

$K = \{ x \mid Ax \geq 0, x \in R^n \}$.

$K^* = \{ A^T \theta \mid \theta \geq 0 \}$ Ex: $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$

$(K^*)^* = K$

$K^* \neq K$ ← may not be equal.