2. Convex Set: Terms and Definitions

Definitions: I. Affine Set, II. Cone, and III. Convex Hull

Given $u_1, u_2, \cdots, u_k \in R^n$,

function $f(u, \theta) = \theta_1 u_1 + \theta_2 u_2 + \cdots + \theta_k u_k$ $\quad \theta_i \in R$

and two conditions

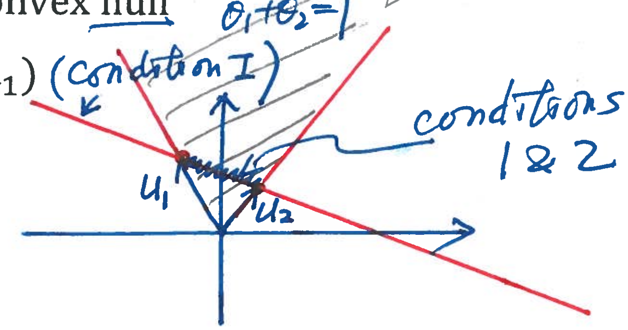
1. $\theta_1 + \theta_2 + \cdots + \theta_k = 1$
2. $\theta_i \geq 0 \ \forall i$

I. $\{f(u,\theta) \mid \text{condition 1}\}$: Affine set $\quad$ min convex set that contains all elements u_i

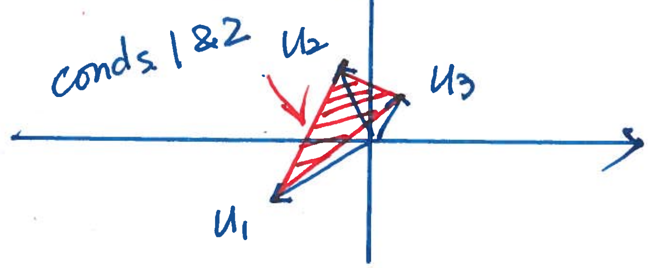
II. $\{f(u,\theta) \mid \text{condition 2}\}$: Cone

III. $\{f(u,\theta) \mid \text{conditions 1 and 2}\}$: Convex hull

$Ex1: \theta_1 u_1 + \theta_2 u_2 = u_1 + \theta_2(u_2 - u_1)$ (condition 1)



$Ex2: \theta_1 u_1 + \theta_2 u_2 + \theta_3 u_3$



2. Sets and Definitions: VI. Hyperplanes and Half Spaces

Hyperplane $\{x \mid a^T x = b\}, a \in R^n, b \in R$ $\quad (a^T x_0 = b)$

or $\{x \mid a^T(x - x_0) = 0\}$, for any $x_0 \in R^n, a \in R^n, b \in R$

Half Space $\{x \mid a^T x \leq b\} \quad a \in R^n, b \in R$

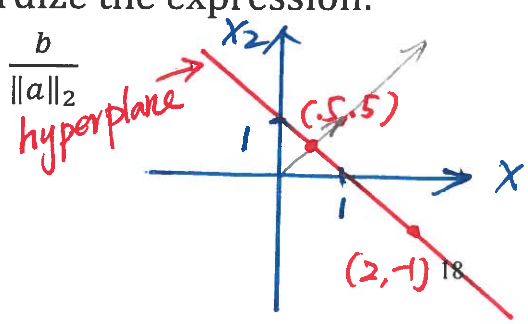
or $\{x \mid a^T(x - x_0) \leq 0\}$

Ex: $\{x \mid x_1 + x_2 = 1\}$ or $\{x \mid [1,1]\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right) = 0\}$

or $\{x \mid a^T(x - x_0) = 0\}$, $\quad a^T = [1,1], b = 1, x_0 = [2, -1]$

For many applications, we standardize the expression:

normalize the expression: $\frac{a^T}{\|a\|_2} x = \frac{b}{\|a\|_2}$



$$\boxed{Ax \le b}$$

$$\begin{cases} a_1^T x \le b_1 \\ a_2^T x \le b_2 \\ \vdots \\ a_m^T x \le b_m \end{cases}$$

$$H = \{ x \mid a^T (x - x_0) = 0 \}$$
$x \in R^n$, $x_0 \in H$

(1). Degree of freedom H : $n-1$

(Simplest nontrivial) exp of R^n space.

(2). H separate the space into 3 parts.

$$S_1 = \{ x \mid a^T (x - x_0) > 0 \}$$

$$S_2 = \{ x \mid a^T (x - x_0) < 0 \}$$

$H \cup S_1 \cup S_2$ whole space.

(3). H, S_1, S_2 all are convex

(4). $\forall\, y \in H$. $a^T (y - x_0) = 0$.

$\forall\, y \in S_1$, $a^T (y - x_0) > 0$.

$\forall\, y \in S_2$, $a^T (y - x_0) < 0$.

(5). $\tilde{H} = \{ x \mid \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} x = 0, x \in R^n \}$

$S_4 = \{ x \mid \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} x < 0, x \in R^n \}$

Cone: $S_5 = \{ [a_1\ a_2] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \mid \theta_1, \theta_2 \ge 0 \}$

Given a $y \in S_5$, $y^T x \le 0$, $\forall x \in \cancel{\tilde{H}}\ S_4$

Proof $\left([a_1, a_2] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \right)^T x = [\theta_1\ \theta_2] \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} x = \theta_1 a_1^T x + \theta_2 a_2^T x < 0$

Affine Set $\overset{S_1}{\{\theta_1 u_1 + \theta_2 u_2 + \cdots \theta_k u_k \mid \Sigma\theta_i = 1\}}$ $\overset{S_2}{\{x \mid \underset{m\times n}{A} x = b\}}_{m<n}$

Given $y_1 \in S_2$, $y_2 \in S_2$

Claim: $\theta_1 y_1 + \theta_2 y_2 \in S_2$ where $\theta_1 + \theta_2 = 1$

Proof

$$A(\theta_1 y_1 + \theta_2 y_2) = \theta_1 A y_1 + \theta_2 A y_2$$
$$= \theta_1 b + \theta_2 b = (\theta_1 + \theta_2) b = b$$

Programming about affine set

$\min \; c^T x \qquad x \in R^n$

$d^T x = b \qquad c, d \in R^n,\; b \in R$.

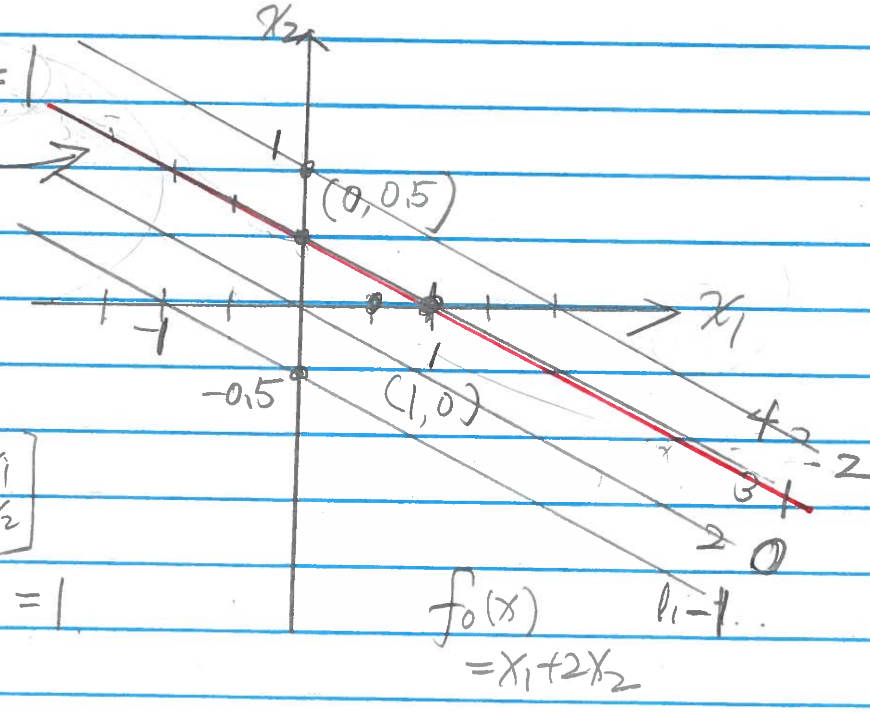
Ex1: $\min \; x_1$

$x_1 + 2x_2 = 1$

$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

hyperplane

affine set.

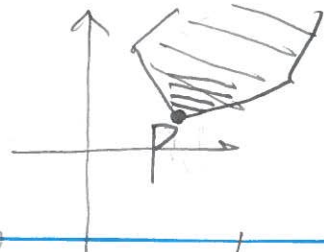


$-\infty$

Ex.2 $\min \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

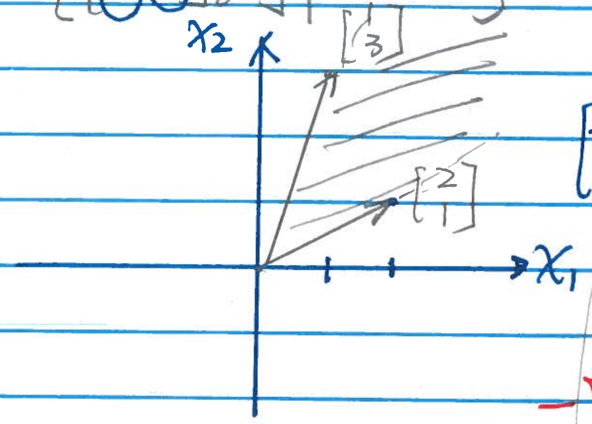
$f_0(x) = x_1 + 2x_2 = 1$

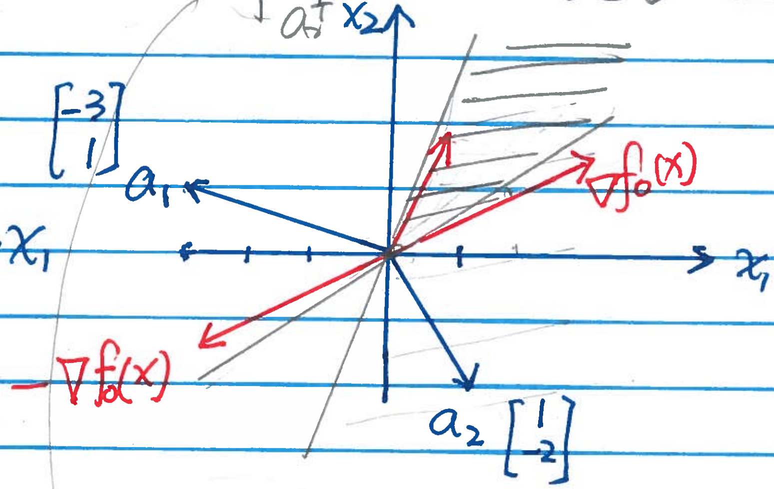
$\nabla f_0(x)$



Cone $\underbrace{\{U\theta \mid \theta_i \ge 0\ \forall i\}}_{S_1}$ $\qquad$ $\underbrace{\{x \mid Ax \le 0,\ x \in R^n\}}_{S_2}$

Ex: $\left\{\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} \middle| \theta_1, \theta_2 \ge 0\right\}$ $\qquad$ $\left\{\begin{bmatrix}x_1\\ x_2\end{bmatrix} \middle| \begin{array}{c|c} a_1^T & \\ a_2^T & \end{array}\begin{bmatrix}-3 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \le \begin{bmatrix}0\\ 0\end{bmatrix}\right\}$





$-\nabla f_0(x)$

Dual cone T: $\left\{\begin{bmatrix}-3 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} \middle| \theta_1, \theta_2 \ge 0\right\}$

For a $y \in T$, $x^T y \le 0$, $\forall x \in S_2$

If $-\nabla f_0(x) \in T$, point p is an opt solution.

Ex: $\nabla f_0(x)$

① $f_0(x) = c^T x \rightarrow \nabla f_0(x) = c$

② $f_0(x) = x^T A x + b^T x$, $\nabla f_0(x) = Ax + A^T x + b$

③ Given $f_0(x) = f(a_1 x_1 + a_2 x_2 + \cdots)$

$\dfrac{\partial f_0(x)}{\partial x}$, $\dfrac{\partial f(x)_{(i)}}{\partial x_i}$