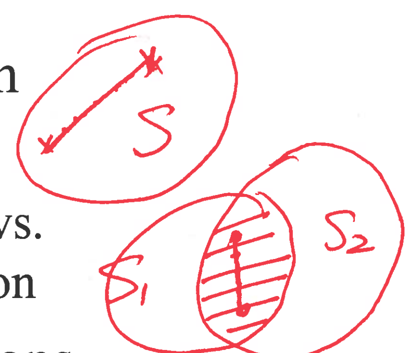
CSE203B Convex Optimization
Lecture 2 Convex Set

CK Cheng
Dept. of Computer Science and Engineering
University of California, San Diego

Chapter 2 Convex Set

1. Set Convexity and Specification
   i. Convexity
   ii. Set Specification: Qualification vs. Enumeration Oriented Description
2. Convex Set Terms and Definitions
3. Separating Hyperplanes
4. Dual Cones



Convex Optimization Problem:

$$\min_{x} f_0(x),\ x \in R^n$$

Subject to

$$f_i(x) \le b_i\ , i = 1, \cdots, m$$

1. $f_0(x)$ is a convex function
2. For $f_i(x) \le b_i,\ i = 1, \dots, m$

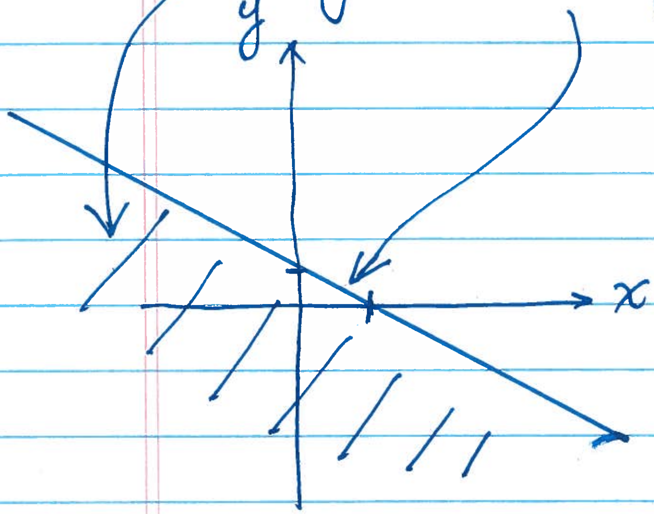
   $\{x | f_i(x) \le b_i, i = 1, \cdots, m\}$ is a convex set

Convex Optimization Problem:

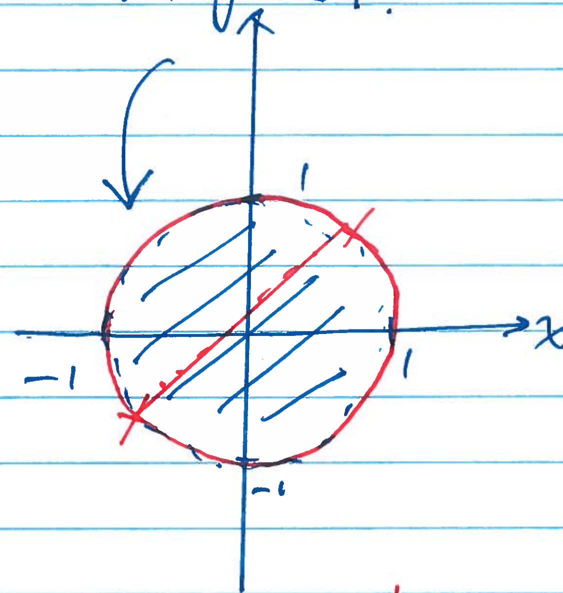
A. Convex Function Definition:

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y), \forall \alpha + \beta = 1, \alpha, \beta \ge 0$$

$x+2y \le 1$. $x+2y=1$

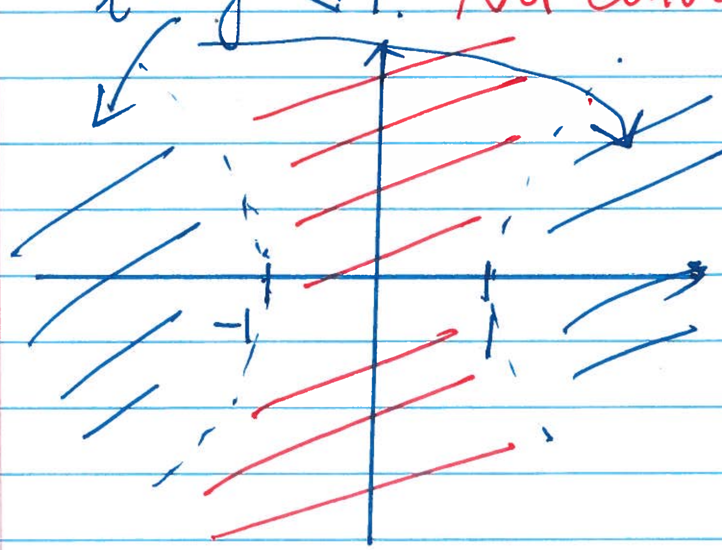


$x^2+y^2 \le 1$.



$x^2+y^2=1$ Not convex.

$x^2-y^2 \le 1$. Not convex.



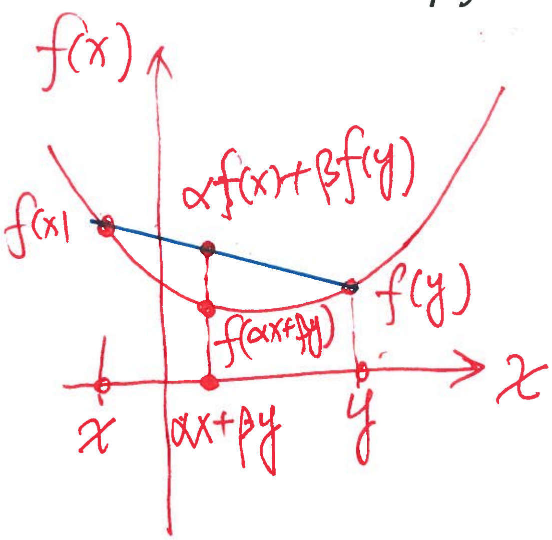
Convex Optimization Problem:

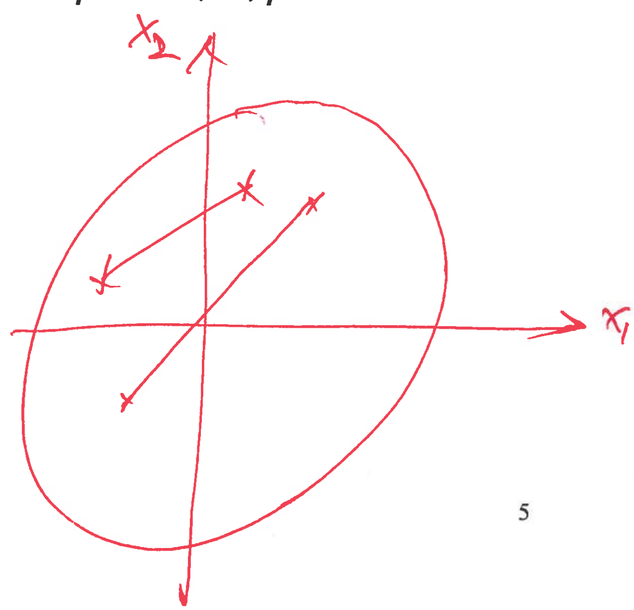
A. Convex Function Definition:

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y), \forall \alpha + \beta = 1, \alpha, \beta \ge 0$$

B. Convex Set Definition: $\forall x, y \in C$

We have $\alpha x + \beta y \in C, \forall \alpha + \beta = 1, \alpha, \beta \ge 0$





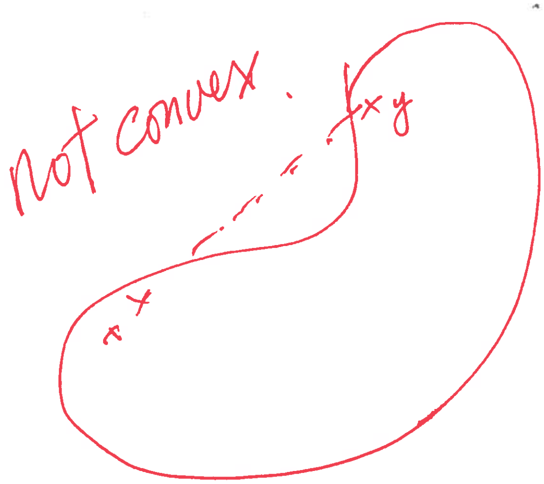
1. Set Convexity and Specification: Convexity

A set S is convex if we have

$\alpha x + \beta y \in S, \forall \alpha + \beta = 1, \alpha, \beta \ge 0, \forall x, y \in S$

Examples:

$x+2y$



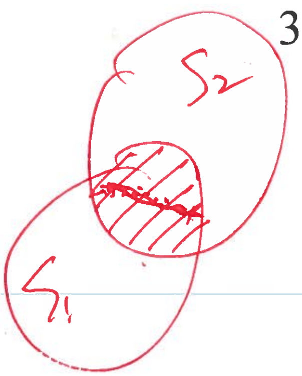
1. Set Convexity and Specification: Convexity

A set S is convex if we have

$$\alpha x+\beta y \in S, \forall \alpha+\beta=1, \alpha, \beta \geq 0, \forall x, y \in S$$

Remark:

1. Most used sets in the class
   1. Scalar set: $S \subset R$
   2. Vector set: $S \subset R^n$
   3. Matrix set: $S \subset R^{n\times m}$
2. Set S is convex if every two points in S has the connected straight segment in the set.
3. For convex sets S_1 and S_2: $S_1 \cap S_2$ is also convex



$$\begin{cases} \forall x,y \in S_1 \Rightarrow (\alpha x+\beta y)\in S_1 \quad \forall \alpha+\beta=1,\ \alpha,\beta\geq 0. \\ \forall x,y \in S_2 \Rightarrow (\alpha x+\beta y)\in S_2 \quad \forall \alpha+\beta=1,\ \alpha,\beta\geq 0. \end{cases}$$

$$\forall x,y \in S_1\cap S_2 \Rightarrow (\alpha x+\beta y)\in S_1\cap S_2,\ \forall \alpha+\beta=1, \alpha,\beta\geq 0.$$

1. Set Convexity and Specification: Set Specification via Qualification or Enumeration

<u>Qualification Oriented Expression</u> $S_Q=\{x \mid Ax \leq b, x \in R^n\}$

<u>Enumeration Oriented Expression</u> $S_E=\{Ax \mid x \in R_+^n\} \in R^m$

$Ax \in R^m \qquad A \in R^{m\times n}$

Qualification Oriented Expression:

Constraints

Min $f_o(x)$

Subject to

$$Ax \leq b, x \in R^n$$

Enumeration Oriented Expression:

Obj function

Min $f_o(Ax), x \in R_+^n$

$$\min_x f_0(x),\ x\in\{x \mid Ax\leq b\}$$

1. Qualification vs Enumeration Oriented Description

Qualification Oriented Expression

Example: $\{x|Ax \leq b\}$

$$
\begin{array}{rrrl}
x_1 & +2x_2 & +3x_3 & \leq 4 \\
2x_1 & -x_2 & & \leq 3 \\
 & x_2 & +x_3 & \leq 5 \\
 & & x_3 & \leq 10
\end{array}
$$

Remark: Simultaneous linear constraints imply AND, Intersection of the constraints

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 10 \end{bmatrix}
$$

1. Qualification vs. Enumeration Oriented Description

$S_1 = \{x|Ax \leq b, x \in R^n\}$ is a convex set

Proof: Given two vectors $u, v \in S_1,\ i.e.\ Au \leq b,\ Av \leq b$

For $w = \theta_1 u + \theta_2 v, \forall \theta_1 + \theta_2 = 1, \theta_1, \theta_2 \geq 0$

we have $Aw \leq b$.

$(Aw = \theta_1 Au + \theta_2 Av \leq \theta_1 b + \theta_2 b = b)$

The inequality implies $w \in S_1$

By definition, set S_1 is convex.

Remark:

1. Simultaneous linear constraints imply AND, Intersection of the constraints
2. Linear constraints constitute a convex set.

$S_2 = \{x \mid Ax \geq b, x \in R^n\}$

$S_3 = \{x \mid Ax = b, x \in R^n\}$

convex

$S_2 = \{x \mid -Ax \leq -b, x \in R^n\}$

convex

1. Qualification vs. Enumeration Oriented Description

Example:

$$S_2 = \{x | Ax \geq b, x \in R^n\}$$

$S_2 = \{x | -Ax \leq -b,\ x \in R^n\}$ Let $\bar{A} = -A$

$\bar{b} = -b$

$$S_3 = \{x | Ax = b, x \in R^n\}$$

$S_2 = \{x | \bar{A}x \leq \bar{b},\ x \in R^n\}$

is convex.

$S_3 = \{x | Ax \leq b,\ -Ax \leq -b,\ x \in R^n\}$

Let $\bar{A} = \begin{bmatrix} A \\ -A \end{bmatrix}$ $\bar{b} = \begin{bmatrix} b \\ -b \end{bmatrix}$

$S_3 = \{x | \bar{A}x \leq \bar{b},\ x \in R^n\}$

1. Qualification Oriented Expression

Example: $S = \{x \in R^m |\ |p_x(t)| \leq 1 \text{ for } |t| \leq \frac{\pi}{3}\}$

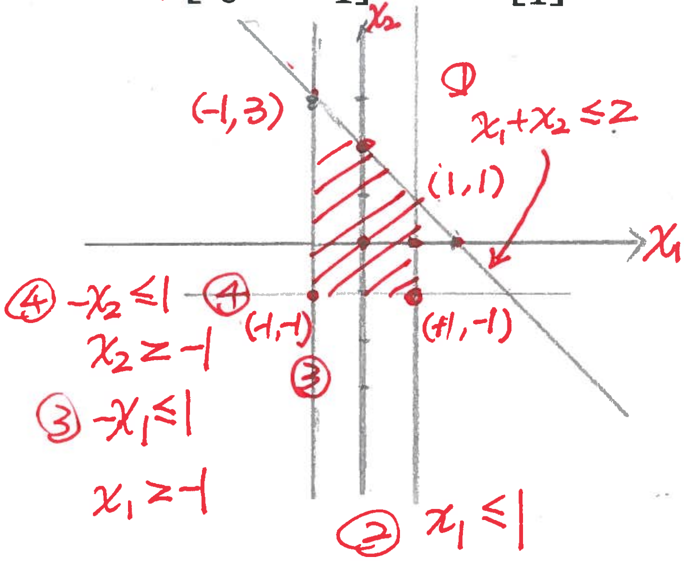
$$where\ p_x(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$$

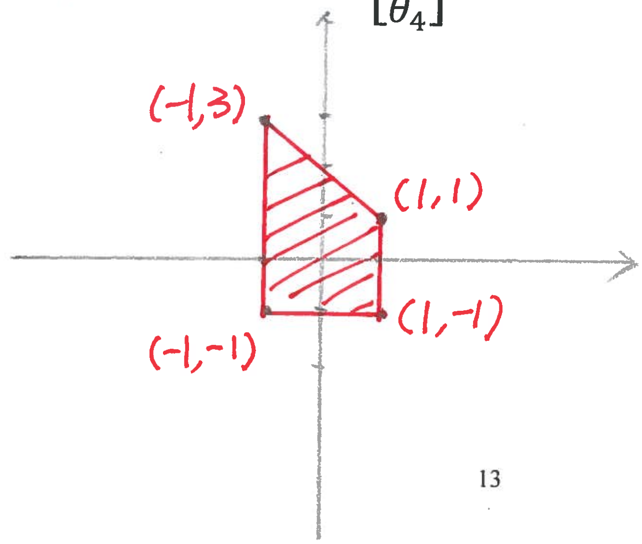
1. Enumeration Oriented Expression

Expression Conversion

Example: $\{x|Ax \leq b, x \in R^n\}$ vs $\{U\theta|\ 1^T\theta = 1,\ \theta \in R_+^m\}$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$





1. Qualification vs. Enumeration Oriented Description

Remark:
Qualification Oriented Expression: Constraints of the problem
Enumeration Oriented Enumeration: The objective function
The interchange may not be trivial.

$A_{p\times n}$

$$\begin{gathered} \min f_0(x) \\ s.t.\ Ax \leq b \quad {\scriptstyle p\times 1} \\ x \in R^n \end{gathered} \qquad\qquad \begin{gathered} \min f_0(U\theta) \\ s.t.\ I^T\theta \leq 1 \\ U \in R^{nm}, \theta \in R_+^m \end{gathered}$$

Every vector u_i in matrix U is a solution of
n equations in constraint $Ax \leq b$

p equations → *comb(p, n) possible*
n variables   *vertex points.*

1. Qualification vs. Enumeration Oriented Description

Mixed Description

$$S_4 = \left\{ \frac{Ax+b}{c^Tx+d} \middle| (c^Tx+d) > 0, x \in C_4 \right\} \text{ (Projective Function)}$$

$$S_5 = \left\{ \frac{z}{t} \middle| z \in R^n, t > 0, (z,t) \in C_5 \right\} \text{ (Perspective Function)}$$

S_4 is convex if C_4 is convex
S_5 is convex if C_5 is convex

1. Qualification vs. Enumeration Oriented Description

Statement: S_5 *is convex* if C_5 *is convex.*

Proof: Given $\left(\frac{z_1}{t_1}\right) \in S_5$, $\left(\frac{z_2}{t_2}\right) \in S_5$, let us set

$$z_3 = \alpha z_1 + \beta z_2, t_3 = \alpha t_1 + \beta t_2, \forall \alpha + \beta = 1, \alpha, \beta \geq 0$$

We have $$\frac{z_3}{t_3} = \frac{\alpha z_1 + \beta z_2}{\alpha t_1 + \beta t_2} = \frac{\alpha t_1}{\alpha t_1 + \beta t_2}\frac{z_1}{t_1} + \frac{\beta t_2}{\alpha t_1 + \beta t_2}\frac{z_2}{t_2}$$

Let $\alpha' = \frac{\alpha t_1}{\alpha t_1 + \beta t_2}, \beta' = \frac{\beta t_2}{\alpha t_1 + \beta t_2}$

(Note that $\alpha' + \beta' = 1, \alpha', \beta' \geq 0$),

we have $\frac{z_3}{t_3} = \alpha' \frac{z_1}{t_1} + \beta' \frac{z_2}{t_2} \in S_5$

Therefore, by definition S_5 is convex.

Qualification → Enumeration conversion

$$S = \{ x \mid Ax = 0, x \in R^n \} \quad m < n$$
$A \in R^{mn}$

$$= \{ (I - QQ^T)x \mid x \in R^n \} \quad \text{where } \underset{nm}{Q} \underset{mm}{R} = \underset{nm}{A^T}$$
(QR decomposition)

$$= \{ x - Q(Q^T x) \mid x \in R^n \}$$
$\underset{mn}{Q^T} \underset{nm}{Q} = \underset{mm}{I}$ orthonormal matrix

Proof:

$$A(I - QQ^T)x = (A - AQQ^T)x \quad (A = R^T Q^T)$$

$$= (R^T Q^T - R^T Q^T Q Q^T)x$$

$$= (R^T Q^T - R^T Q^T)x$$

$$= 0.$$

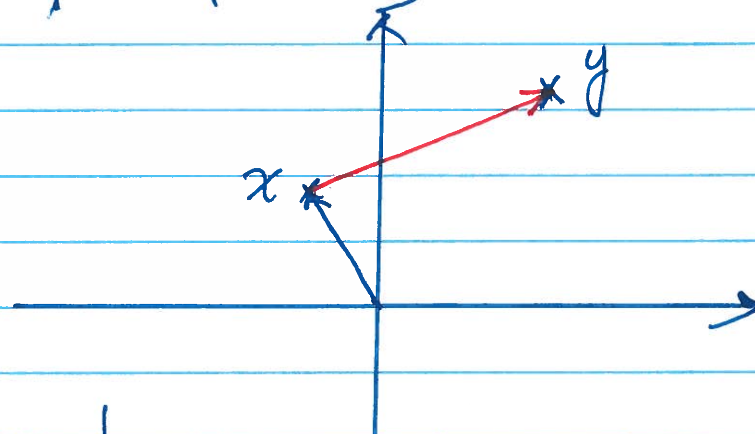
$$S = \left\{ \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} [x_2] \,\middle|\, x_2 \in R^{n-m} \right\}$$

$$[A_1 \; A_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow A_1 x_1 + A_2 x_2 = 0 \Rightarrow x_1 = -A_1^{-1} A_2 x_2.$$

$$S_1 = \{ \alpha x + \beta y \mid \alpha + \beta = 1,\ \alpha, \beta \geq 0 \}$$

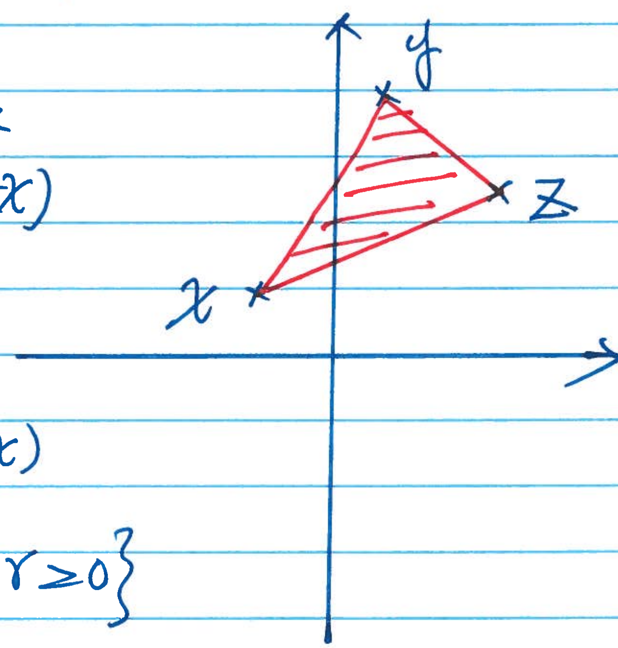
$$\alpha x + \beta y = (1-\beta) x + \beta y = x + \beta (y - x)$$

$$S_1 = \{ \underline{x + \beta (y - x)} \mid 1 \geq \beta \geq 0 \}$$



$$S_2 = \{ \alpha x + \beta y + \gamma z \mid \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma \geq 0 \}$$

$$\begin{aligned} & \alpha x + \beta y + \gamma z \\ &= (1 - \beta - \gamma) x + \beta y + \gamma z \\ &= x + \beta (y - x) + \gamma (z - x) \end{aligned}$$



$$S_2 = \{ x + \beta (y - x) + \gamma (z - x) \mid 1 \geq \beta + \gamma \geq 0,\ \beta, \gamma \geq 0 \}$$

$$S_1 = \{ x \mid \underset{m\times n}{A} x = 0,\ x \in R^n \} \qquad m < n$$

$$m \updownarrow \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad \overset{\longleftrightarrow}{n}$$

$$S_2 = \{ (I - QQ^T) x \mid x \in R^n \}$$

Ad: Preserve the x

Dis: P is a singular matrix.

where

$$\underset{[a_1 \cdots a_m]}{A^T} = \underset{n\times m}{Q}\ \underset{m\times m}{R} \quad \text{QR decomposition}$$

$$Q = [q_1 \cdots q_m]$$

$$\boxed{\underset{m\times n}{Q^T}\ \underset{n\times m}{Q} = \underset{m\times m}{I}}.$$

$P = \underbrace{I - QQ^T}$ To prove that $AP = 0$.

Proof

$$\begin{aligned} AP &= (QR)^T (I - QQ^T) \\ &= R^T Q^T (I - QQ^T) \\ &= R^T Q^T - R^T \boxed{Q^T Q} Q^T \\ &= R^T Q^T - R^T Q^T = 0 \end{aligned}$$

$$A : [\underset{m\times m}{A_1}\ \underset{m(n-m)}{A_2}] \ ; \quad x : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} m\times 1 \\ (n-m)\times 1 \end{matrix}.$$

$$A_1 x_1 + A_2 x_2 = 0. \quad x_1 = -A_1^{-1} A_2 x_2$$

$$S_3 = \left\{ \begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix} x_2 \;\middle|\; x_2 \in R^{n-m} \right\}$$

Ad: Clean conversion

Dis: Create "many" nonzero elements

2. Convex Set: Terms and Definitions

Definitions: I. Affine Set, II. Cone, and III. Convex Hull

Given $u_1, u_2, \cdots, u_k \in R^n$,

function $f(u,\theta) = \theta_1 u_1 + \theta_2 u_2 + \cdots + \theta_k u_k$ $\theta_i \in R$

and two conditions

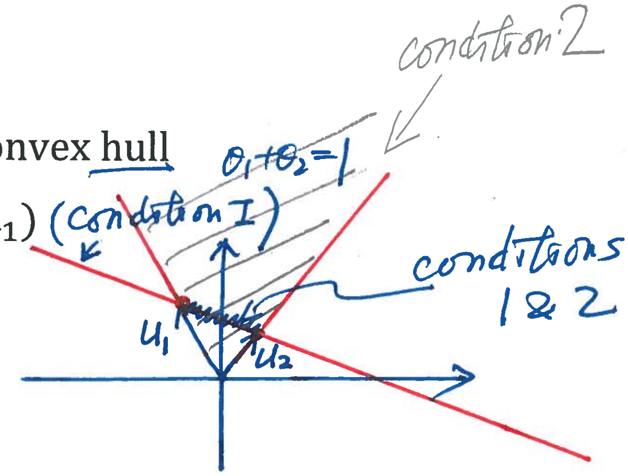
1. $\theta_1 + \theta_2 + \cdots + \theta_k = 1$
2. $\theta_i \geq 0 \ \forall i$

I. $\{f(u,\theta) \mid \text{condition 1}\}$: Affine set

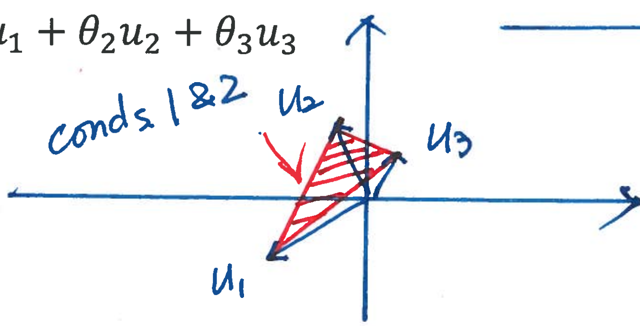
II. $\{f(u,\theta) \mid \text{condition 2}\}$: Cone

III. $\{f(u,\theta) \mid \text{conditions 1 and 2}\}$: Convex hull

*Ex*1: $\theta_1 u_1 + \theta_2 u_2 = u_1 + \theta_2(u_2 - u_1)$ (condition I)



*Ex*2: $\theta_1 u_1 + \theta_2 u_2 + \theta_3 u_3$



2. Sets and Definitions: VI. Hyperplanes and Half Spaces

Hyperplane $\{x \mid a^T x = b\}, a \in R^n, b \in R$ $(a^T x_0 = b)$

or $\{x \mid a^T(x - x_0) = 0\}$, for any $x_0 \in R^n, a \in R^n, b \in R$

Half Space $\{x \mid a^T x \leq b\}$ $a \in R^n, b \in R$

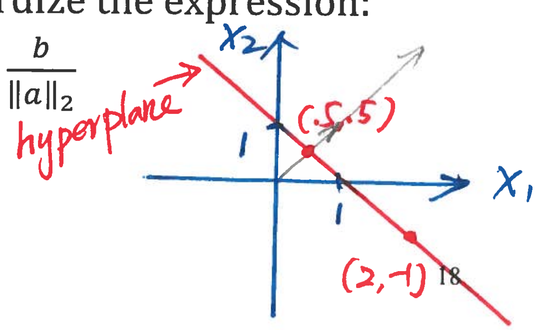
or $\{x \mid a^T(x - x_0) \leq 0\}$

Ex: $\{x \mid x_1 + x_2 = 1\}$ *or* $\{x \mid [1,1]\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right) = 0\}$

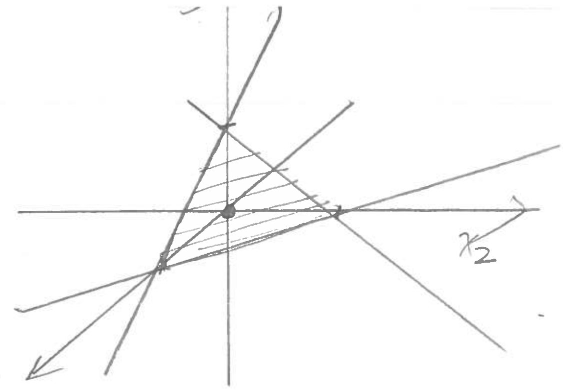
or $\{x \mid a^T(x - x_0) = 0\}$, $a^T = [1,1], b = 1, x_0 = [2,-1]$

For many applications, we standardize the expression:

normalize the expression: $\frac{a^T}{\|a\|_2} x = \frac{b}{\|a\|_2}$



2. Sets and Definitions: Hyperplanes



Ex : 3 variables

$\{x|a^T x = b\},\ a^T = (1,1,1),\ b = 6$

Ex : 4 variables

$\{x|a^T x = b\},\ a^T = (1,1,1,1),\ b = 6$

(1) degrees of freedom : $n-1$ (R^n).

(2) all vectors $(x-y)$ are orthogonal to direction a, i.e.
$a^T(x-y) = 0,\ \forall x, y$ in the hyperplane

Proof: $H = \{x \mid a^T(x - x_0) = 0\}$ where $a^T x_0 = b$

(1) $x_0 \in H \Rightarrow a^T(x - x_0) = 0 \ \forall x \in H$.

Exercise: Conceptually (visually) construct hyperplane.

2. Sets and Definitions: Hyperplanes

Hyperplane : as an Equal potential of cost function

$$\min f_0(x) = c^T x$$

$$e.g.\ [1,2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{\partial f_0(x)}{\partial x_1} = 1$$

$$\frac{\partial f_0(x)}{\partial x_2} = 2$$

Vector c is the sensitivity or cost of vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$