CSE203B Convex Optimization
Chapter 11 Interior Point Methods

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Chapter 11: Interior-Point Methods

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- Formulation
  - Inequality constrained optimization
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Introduction

- Frisch, R., 1956. La résolution des problèmes de programme linéaire par la méthode du potentiel logarithmique. Cahiers du Seminaire D'Econometrie, pp.7-23.
- Dikin, I.I., 1967. Iterative solution of problems of linear and quadratic programming. In Doklady Akademii Nauk (Vol. 174, No. 4, pp. 747-748). Russian Academy of Sciences.
- Karmarkar, N., 1984, December. A new polynomial-time algorithm for linear programming. In Proceedings of the sixteenth annual ACM symposium on Theory of computing (pp. 302-311).
- Wright, M., 2005. The interior-point revolution in optimization: history, recent developments, and lasting consequences. Bulletin of the American mathematical society, 42(1), pp.39-56.
- Nesterov, Y. and Nemirovskii, A., 1994. Interior-point polynomial algorithms in convex programming. Society for industrial and applied mathematics.

Formulation: The problem

Softmax, $\max\{x_1, x_2 \cdots x_m\}$

$\ln \sum_i e^{\gamma x_i}$

Problem: $\min f_0(x)$

Subject to $f_i \leq 0,\ i = 1, \dots, m$

$$Ax = b$$

Function f_is are convex, twice continuously differentiable

We assume that rank $A = p, A \in R^{p \times n}$.

Issues:

- KKT conditions on inequality constraints
  - $\lambda_i = 0$, if $f_i(x) < 0$; otherwise $\lambda_i > 0$ $(\lambda_i f_i(x) = 0)$
- m can be large.
- When to put $f_i = 0$ (active)? There are 2^m combination.

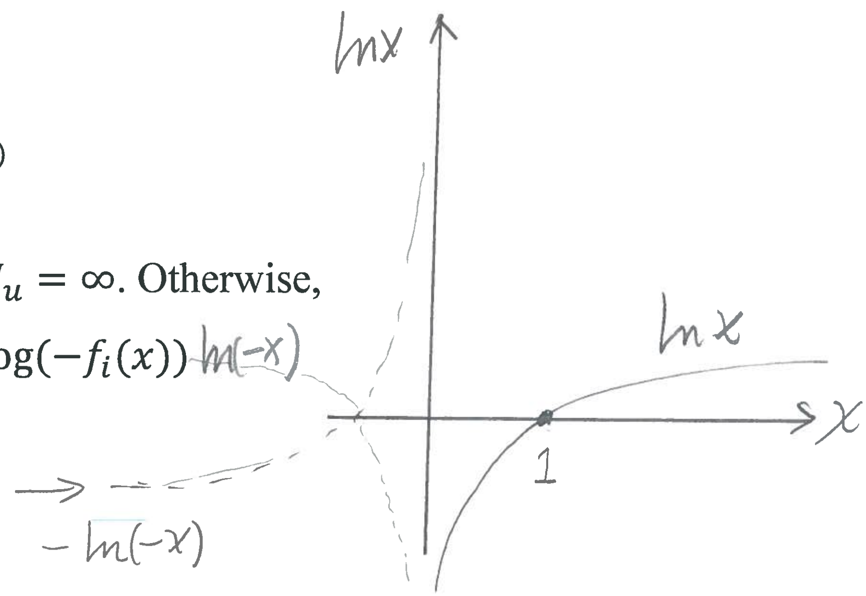
Formulation: logarithmic barrier

Problem:

$$\min f_0(x) + \sum_{i=1}^{m} I_{f_i(x)}$$
$$s.t. \quad Ax = b$$

When $I_u = 0$ if $u \le 0$, $I_u = \infty$. Otherwise,

$$min\ f_0(x) + \frac{-1}{t}\sum_{i=1}^{m} \log(-f_i(x))$$
$$s.t. \quad Ax = b$$



Remark:

1. Convert inequality constraints to barrier functions.
2. Incorporate barrier functions in objective function.
3. Increase t to improve accuracy.

Formulation: logarithmic barrier

Let us set

$$\phi(x) = -\sum_{i=1}^{m} log(-f_i(x)), \qquad dom\ \phi = \{x | f_i(x) < 0\}$$

$\phi(x)$ is convex and twice differentiable

$$\nabla\phi(x) = \sum_{i=1}^{m} -\frac{1}{f_i(x)}\nabla f_i(x)$$

$$\nabla^2\phi(x) = \sum_{i=1}^{m} \frac{1}{f_i(x)^2}\nabla f_i(x)\nabla f_i(x)^T - \frac{1}{f_i(x)}\nabla^2 f_i(x)$$

Central Path is $\{x^*(t) | t > 0\}$

$$min \quad tf_0(x) + \phi(x)$$
$$s.t. \quad Ax = b$$

Formulation: logarithmic barrier

Ex:

Problem: $\min \quad c^T x$ $\quad f_i(x) = a_i^T x - b_i \le 0$

$s.t. \quad a_i^T x \le b_i, \qquad i = i, \dots, m$

Log barrier formulation:

$$\min tc^T x - \sum_{i=1}^{m} \log(b_i - a_i^T x)$$

Hyperplane $c^T x = c^T x^*(t)$ *is tangent to real curve* φ *through* $x^*(t)$.

Solution $x^*(t)$ *balance the force between* $-t\nabla f_0(x)$ *and* $\sum_{i=1}^{m} -\frac{1}{f_i(x)} \nabla f_i(x)$.

Formulation: logarithmic barrier

Ex:

Problem: $\min \quad c^T x$

$s.t. \quad a_i^T x \le b_i \qquad i = i, \dots, m$

$-t\nabla f_0(x) = -tc$

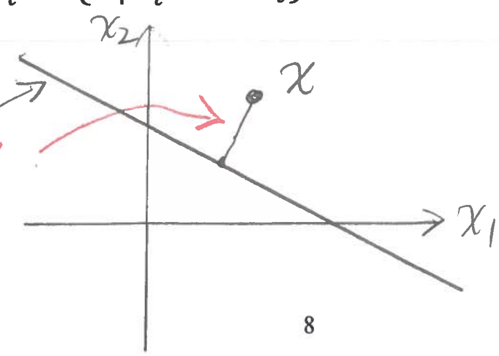
$P(x) = c^T x - \sum_i \ln(b_i - a_i^T x)$

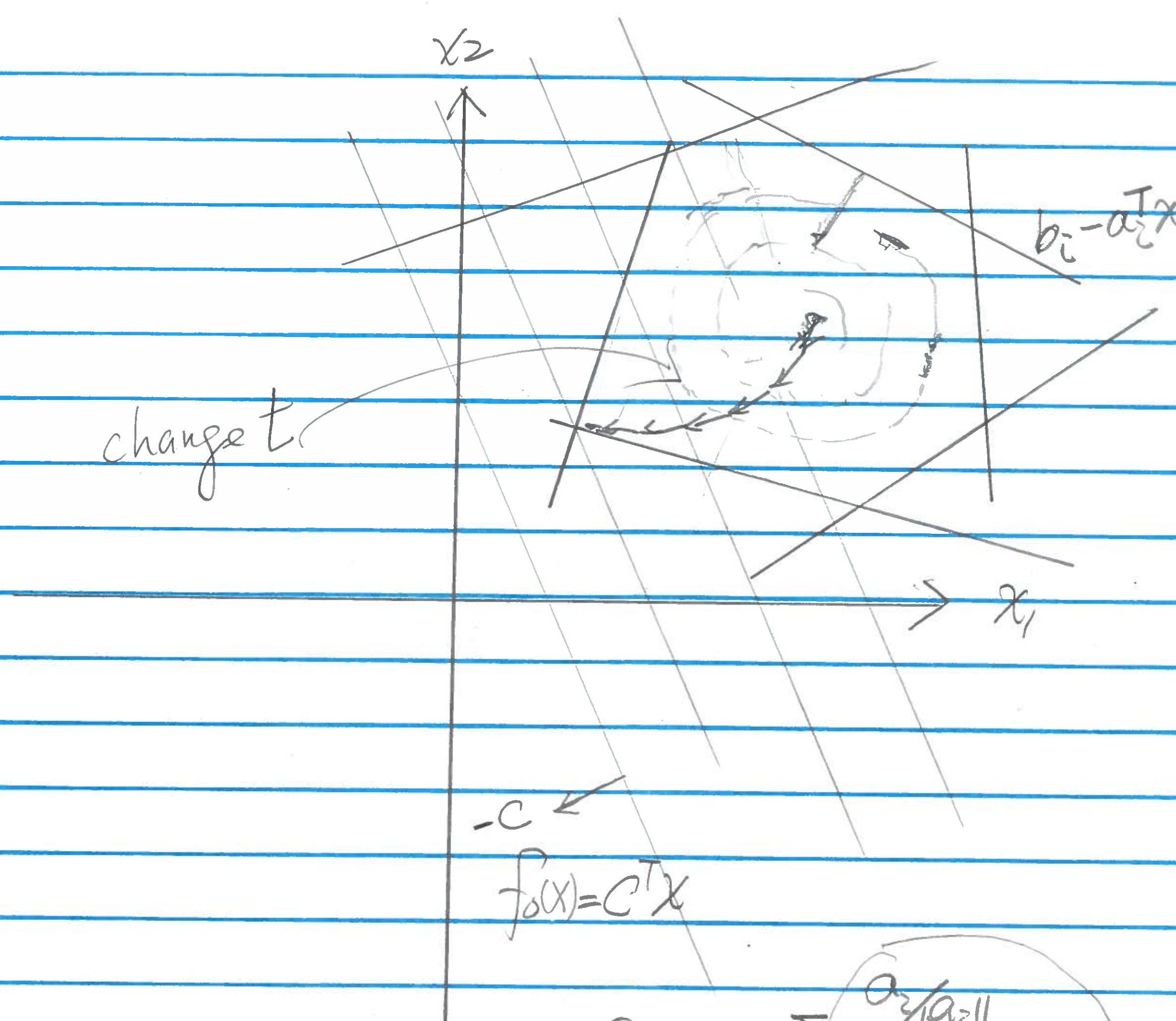
$\nabla P(x) = c - \sum_i \frac{a_i}{b_i - a_i^T x} = c - \sum_i \frac{a_i/\|a_i\|}{\frac{b_i}{\|a_i\|} - \frac{a_i^T}{\|a_i\|} x}$

$$\sum_{i=1}^{m} -\frac{1}{f_i(x)} \nabla f_i(x) = \sum_{i=1}^{m} -\frac{1}{b_i - a_i^T x} a_i$$

Note that $\min \left\| \frac{1}{b_i - a_i^T x} a_i \right\|_2 = \frac{1}{dist(x_i H_i)}$, $H_i = \{x | a_i^T x = b_i\}$

$\frac{b_i}{\|a_i\|} - \frac{a_i^T}{\|a_i\|} x$





$$c \qquad \sum_i \frac{a_i/\|a_i\|}{\frac{b_i}{\|a_i\|} - \frac{a_i^T}{\|a_i\|}x}$$

$$\text{Obj}: P(x) = t c^T x - \sum_i \ln(b_i - a_i^T x)$$

$$\nabla P(x) \Rightarrow t c$$