
On Optimization of Isospectralization

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Abstract

Whether one can recover the shape of an object through its Laplacian spectrum is a classical problem with a wide range of applications on computer vision and graphics. While theoretically we cannot fully recover the shape only based on spectrum, in practice we can approximate the solution. In this report we introduce a specific solution, namely isospectralization. We investigate its mathematical background, implement the procedure using modern deep learning framework, and reformulate a novel dual problem solved by projected subgradient method.

1 Introduction

Even long before the French mathematician Joseph Fourier discovered Fourier theorem have people already been quite familiar with the relation between the pitch of a sound and its frequency of vibration. With the relation in mind, people have been constructing musical instruments by strings for thousands of years, also aware of the fact that the frequency is closely related to the length of a string. An interesting byproduct of this knowledge is that given two strings tightened to a similar level, all the same except for lengths, one can tell the relative length by the sounds from plucking them. Can these abilities be generalized to other objects such as 2D drum and even 3D object? In other words, can we tell the shape of an object by hearing the sound of it? Mathematician and computer scientists have been studies on this problem for decades. Although exact recovery is impossible, approximation algorithms work quite well. We will introduce mathematical background, formal problem description, a relevant algorithm and its potential improvement for this problem.

2 Mathematical Background

2.1 Wave Equation and the Laplacian

We will introduce how the case of one dimensional string can be extended into two dimensions. Given a membrane on a 2D plane (with x-y Cartesian coordinates) fixed along its boundary Γ , its motion perpendicular to the plane is called displacement, denoted as $F(x, y; t) \equiv F(\vec{p}; t)$. The displacement must follow the wave equation

$$\frac{\partial^2 F}{\partial x^2} = c^2 \Delta F$$

with boundary condition $F = 0$. In this equation Δ is the Laplacian operator and c is some constant related to the physical properties of the membrane; for simplicity, let $c^2 = \frac{1}{2}$. With the wave being sound wave, the solution to this wave equation must be of special form (being harmonic in time): $F(\vec{p}; t) = U(\vec{p})e^{i\omega t}$. After some derivation the solutions must satisfy $\frac{1}{2}\Delta U + \omega^2 U = 0$ with the aforementioned boundary condition.

2.2 Laplacian Spectrum

After decades of efforts, mathematicians discovered that there exists a set of solution for the equation above, namely, a set of eigenfunction $\psi_n(\vec{p}) \equiv \psi_n$ such that with the corresponding eigenvalues λ_n

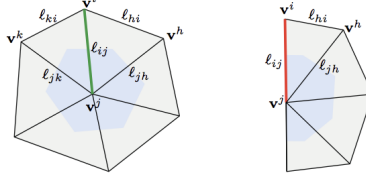


Figure 1: Edge e_{ij} has length l_{ij} , triangle F_{ijk} has area A_{ijk} .

the following equations hold:

$$\frac{1}{2}\Delta\psi_n + \lambda_n\psi_n = 0$$

Similar to the case in linear algebra, the set of eigenfunctions and eigenvalues are called the spectrum of the Laplacian operator.

The problem of "hear" the shape of the membrane is then articulated as to compute the boundary of the membrane given its Laplacian spectrum, called **isospectralization**, the term in the title. Notice that in practice problems are discretized, in the sense that we only try to recover the discrete structure of the membrane, represented by a mesh (or a weighted graph). Accordingly the Laplacian would be a discrete Laplace operator of the underlying graph. The details of this process will be introduced in section 3.

2.3 Calculus of Spectrum

Since we deal with isospectralization in an optimization approach, the process involves computing the gradient of the spectrum. The derivation is as below, with the fact that the (discrete) Laplacians are always Hermitian. Given normalized eigenvector v of the matrix A such that $Av = \lambda v$, if differentiate two sides of the equation, we have

$$\partial(A)v + A\partial(v) = \partial(\lambda)v + \lambda\partial v$$

$$v^T\partial(A)v + v^T A\partial(v) = v^T\partial(A)v = \partial\lambda$$

Notice that the calculation involves calculating the eigenvectors; this might lead to some inefficiency in optimization based approach. Because if the eigen-gap of the Laplacian is relatively small, the numerical errors caused in computing the eigenvectors might be devastatingly large, leading to inaccurate gradient.

3 Problem Formulation

3.1 Laplacian of Discrete Triangle Mesh

In the discrete setting, one shape can be approximated by manifold triangle meshes $X = (V, E, F)$, where V is the set of vertices sampled at v_1, \dots, v_n , and where each edge $e_{ij} \in E_i \cap E_b$ belongs to at most two triangle faces F_{ijk} and F_{jih} , as shown in Figure 1. We use E_i and E_b to denote the interior and boundary edges respectively. The discrete metric is defined by assigning a length $l_{ij} > 0$ to each edge $e_{ij} \in E$. Here we use L2 distance: $l_{ij}(V) = \|v_i - v_j\|_2$ for all $e_{ij} \in E$. The discrete Laplace-Beltrami operator is defined in the form of a n by n matrix $\Delta = A^{-1}W$, where A is a diagonal matrix of local area elements $a_i = \sum_{jk:i,j,k \in F} A_{ijk}/3$, and W is a symmetric matrix of edge-wise weights, defined as:

$$w_{ij} = \begin{cases} \frac{-l_{ij}^2 + l_{jk}^2 + l_{ki}^2}{8A_{ijk}} + \frac{-l_{ij}^2 + l_{jh}^2 + l_{hi}^2}{8A_{jih}} & e_{ij} \in E_i \\ \frac{-l_{ij}^2 + l_{jh}^2 + l_{hi}^2}{8A_{jih}} & e_{ij} \in E_b \\ -\sum_{i \neq k} w_{ik} & i = j \end{cases}$$

The mesh connectivity and intrinsic geometry are all encoded in this Laplacian operator, which is related with the vertex coordinates V (via the lengths l_{ij}). We further denote the Laplacian matrix of the mesh of V by $\Delta_X(V)$.

3.2 Isospectralization

Our approach focuses on how to use a limited portion of Laplacian spectrum to reconstruct the shape of the target, with some simple regularizers ρ to preserve the boundary or triangulized properties. Given an initialization of coordinates V , and a limited portion of Laplacian spectrum μ_i from the target, we want to minimize the L2 distance of $\lambda_i(\Delta_X(V))$ and μ_i . The objective is:

$$\min_{V \in \mathbb{R}^{n \times d}} \|\lambda(\Delta_X(V)) - \mu\|_w + \rho(V)$$

where $\|\lambda(\Delta_X(V)) - \mu\|_w = \sum_{i=1}^k \frac{1}{i} (\lambda_i - \mu_i)$, with a scale factor $1/i$ on corresponding eigenvalues, here $\lambda_1 \leq \lambda_2 \dots \leq \lambda_k$. Without this weight, simply aligning the high frequencies will produce smaller loss, but most errors will accumulate on the lower end and harm the result. We thus adopt the weighted norm to balance the effects. To preserve the smoothness of the boundary, we define the first regularizer:

$$\rho_{1,X}(V) = \sum_{e_{ij} \in E_b} l_{ij}^2(V)$$

The second regularizer penalizes triangle flips that may occur throughout the optimization, and works under the assumption of clockwise oriented triangle.

$$\rho_{2,X}(V) = (\sum_{ijk} (\mathbf{R}_{\pi/2}(V_j - V_i))^T (V_k - V_i))_-$$

where $\mathbf{R}_{\pi/2}$ is a rotation matrix rotating vectors by $\pi/2$, and $(x)_- = \min(x, 0)^2$.

3.3 Dual Problem and Projected Subgradient

We usually need to adjust the hyper-parameter added on the regularizers, which is usually quite tricky, since the result can be very sensitive to it. But with some prior knowledge when design this problem, we can accept all the pareto-optimal solutions with regularizers bounded by ϵ . We can reformulate the problem as:

$$\begin{aligned} & \text{minimize}_{V \in \mathbb{R}^{n \times d}} \|\lambda'(\Delta_X(V)) - \mu\|_w \\ & \text{s.t. } \rho(V) \leq \epsilon \end{aligned}$$

Note that in our objective, this regularizer won't introduce convexity to make the optimization easier, so moving it from the objective to constraint won't introduce extra cost. To distinguish between the eigenvalues and Lagrangian dual variable, here we use λ' to denote the eigenvalues of Laplacian Matrix $\Delta_X(V)$. Then we can obtain the Lagrangian of the objective:

$$L(V, \lambda) = \|\lambda'(\Delta_X(V)) - \mu\|_w + \lambda(\rho(V) - \epsilon)$$

Based on the knowledge of Lagrangian dual, we know the objective for the dual problem is

$$g(\lambda) = \inf_V L(V, \lambda) = \inf_V (\|\lambda'(\Delta_X(V)) - \mu\|_w + \lambda(\rho(V) - \epsilon))$$

Here we know the formulation of $L(V, \lambda)$, which is obviously not convex. But for $\inf_V L(V, \lambda)$ when λ is given, we can always find corresponding optimal $V = V^*(\lambda)$ to minimize the Lagrangian. Then by definition, we have:

$$g(\lambda) = (\|\lambda'(\Delta_X(V^*(\lambda))) - \mu\|_w + \lambda(\rho(V^*(\lambda)) - \epsilon))$$

which is a function of λ .

Then we can write the dual problem formulation:

$$\text{max. } g(\lambda) \quad \text{s.t. } \lambda \geq 0$$

Recall the primal problem:

$$\begin{aligned} & \text{minimize}_{V \in \mathbb{R}^{n \times d}} \|\lambda'(\Delta_X(V)) - \mu\|_w \\ & \text{s.t. } \rho(V) \leq \epsilon \end{aligned}$$

Note that if want to optimize the objective in primal, following the update rule $V_{k+1} = \text{Proj}(V_k - \alpha_k * g_k)$, we need to project V_{k+1} to the feasible set every update step. Here g_k is the subgradient (or quasi-gradient) of the objective at V_k , since the objective is not differentiable everywhere for the discrete mesh.

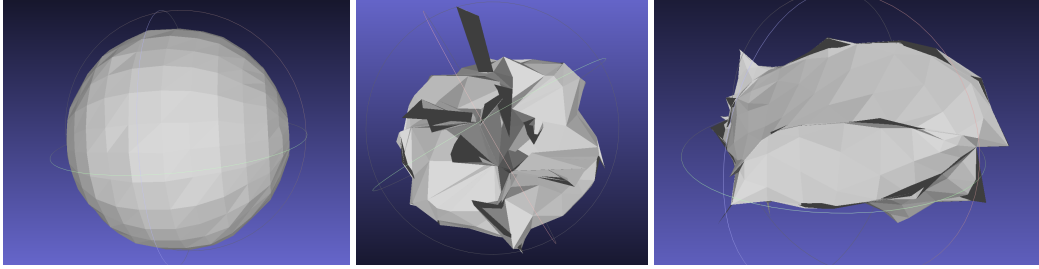


Figure 2: From sphere to apple and airplane in 1000 iterations

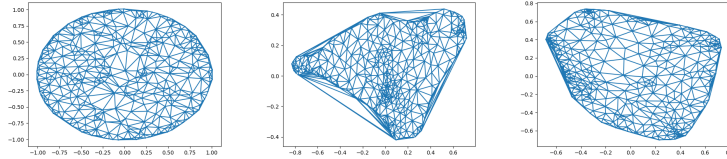


Figure 3: From oval to triangle and rectangle in 1000 iterations

Calculating such projection in primal is infeasible here, since the regularizer is not convex. But for the dual problem, the update rule is: $\lambda_{k+1} = (\lambda_k + \alpha_k * g_k)_+$

The projection in the dual is very easy to compute: just making all the negative elements of λ_{k+1} become zero, which equals to project it to a non-negative convex cone. Since

$$g(\lambda) = \text{inf}_V L(V, \lambda) = \text{inf}_V (\|\lambda'(\Delta_X(V)) - \mu\|_w + \lambda(\rho(V) - \epsilon))$$

The subgradient at λ_k is also easy to compute, which is $g_k = \rho(V^*(\lambda_k)) - \epsilon$.

Suppose $V^*(\lambda_k)$ is an minimizer of Lagrangian given λ_k solved by Adam optimizer. So we can update iteratively in primal and dual.

$$V_k = V^*(\lambda_k), \quad \lambda_{k+1} = (\lambda_k + \alpha_k * (\rho(V_k) - \epsilon))_+$$

This problem will converge both in primal and dual if Slater's condition is satisfied.

4 Experiments

For both 2D and 3D meshes, we only use the largest 30 eigenvalues to reconstruct the shape of the target. For 3D meshes, such number of eigenvalues is not sufficient to reconstruct the shape since the Laplacian have thousands dimensions, and increasing the number will cause the process to be very slow. The reconstruction in 1000 iterations is shown in Figure 2: As for 2D meshes, we show the result from circle to rectangle, and circle to triangle in 1000 iterations in Figure 3.

Individual Contribution

The slides, presentation and report are finished in teamwork. **Jiayuan Gu** Implemented the pipeline of isospectralization using PyTorch, conducted experiments on 2D and 3D meshes. **Zhiwei Jia** Investigate the mathematical background, using Matlab to optimize the L2 distance between eigenvalues and make the result more accurate. **Fangchen Liu** Formulate and analyze the constrained primal and dual problem, using the projected subgradient method to solve the dual problem implemented by PyTorch.

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