CSE 203B W21 Homework 4

Due Time : 11:50pm, Wednesday Feb. 16, 2022 Submit to Gradescope

In this homework, we work on exercies from text book. Problem 4.1, 4.8, 4.11, and 4.15 are related to LP. Problem 4.21, 4.39, and 4.47 are related to QCQP, and SDP. Also, we practice using the convex optimization tools on a linear programming problem and a quadraticallly constrained quadratic programming problems.

Total points: 30. Exercises are graded by completion, assignments are graded by correctness.

I. Exercises from textbook chapter 4 (7 pts, 1pt for each problem)

4.1, 4.8, 4.11, 4.15, 4.21, 4.39, 4.47.

II. Assignments (23 pts)

II.1 Linear Programming: You are free to use any software packages. (10 pts)

Given

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & -1 & 2 \end{bmatrix},$$

$$b^T = \begin{bmatrix} -1 & 2 & 3 & -4 \end{bmatrix},$$

$$c^T = \begin{bmatrix} 1 & -2 & 1 & -1 \end{bmatrix},$$

and $n = 4$, solve the following linear programming problems. If a solution is found, validate that the solution satisfies the optimality criteria (which was talked about in class or textbook). Otherwise, explain why a solution is not feasible and suggest how to mitigate the issue if you are the project leader:

II.1.1. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n$.

II.1.2. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n$.

II.1.3. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n_+$.

II.1.4. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n_+$.

II.2 Graph embedding (13 pts) Graph embedding is an important problem in machine learning and graph theory. Given an undirected graph $G = (V, E)$ with n vertices, the problem is to assign coordinates in $\mathbb{R}^m$ to each vertex $v \in V$. Typically there are desired qualities or constraints imposed on the embedding—e.g. the coordinates assigned to connected nodes should be close with respect to some distance metric. We can formulate this as a quadratically constrained quadratic program (QCQP). Let $A \in \{0,1\}^{n\times n}$ be the symmetric adjacency of G, and let D be the corresponding diagonal degree matrix such that $D_{ii} = \sum_j A_{i,j}$. The *graph Laplacian* is defined to be $L = D - A$.

Let $x, y \in \mathbb{R}^n$ represnt the x and y coordinates of n vertices. Given a parameter $c \in \mathbb{R}_{++}$, one way to define the graph embedding problem in 2-d is to solve the following problem:

$$\begin{aligned} \min_{x,y\in\mathbb{R}^n} \quad & x^\top Lx + y^\top Ly \\ \text{s.t.} \quad & x^\top x \leq c, \quad y^\top y \leq c \end{aligned} \tag{1}$$

(i) Show that $x^\top Lx = \sum_{i,j\in E}(x_i - x_j)^2$
(ii) Consider a partitioning of x; $x = [x_1 : x_2]^\top$, where $x_1 \in \mathbb{R}^{n-k}$ corresponds to the coordinates of $n-k$ "free" nodes and $x_2 \in \mathbb{R}^k$ are the coordinates of k "fixed"/"anchor" nodes (likewise for y). Under these "fixed-node" constraints, show that Prob. 1 is equivalent to

$$\begin{aligned}\min_{x_1,y_1\in\mathbb{R}^{n-k}} \quad & x_1^\top L' x_1 + y_1^\top L' y_1 + b^\top x_1 + d^\top y_1 \\ \text{s.t.} \quad & x_1^\top x_1 \le c'_x, \quad y_1^\top y_1 \le c'_y\end{aligned}$$

In other words, express L', b, d, and c'_x & c'_y in terms of x_1, x_2, y_1, y_2, L, and c. Are there any issues with Prob. 1 if there are no fixed nodes?
(iii) Implement the problem in CVX/CVXPY and show your result for the given graph with $c'_x = c'_y = 10$.
(iii.a) We have written a partial framework in Python to get you started:
`https://colab.research.google.com/drive/1apgxNJGN1E4_W6awYbbhNxTyL0VvvMVH?usp=sharing`.
(iii.b) If you prefer a different language, you can also download a .txt file containing L, x, y, and the indices of the fixed nodes:
`https://piazza.com/class_profile/get_resource/kx85xrdgigl5m5/kzfw6ud6fd964c` (idx, x, y are the first 3 columns)
(iv) Suppose we change the quadratic inequality constraints on x and y to equality constraints and add a constraint $x^\top y = c'_{xy}$. Is the problem still convex? If not, can we still recover a solution?