CSE 203B W21 Homework 4

Due Time : 11:50pm, Wednesday Feb. 16, 2022 Submit to Gradescope

In this homework, we work on exercies from text book. Problem 4.1, 4.8, 4.11, and 4.15 are related to LP. Problem 4.21, 4.39, and 4.47 are related to QCQP, and SDP. Also, we practice using the convex optimization tools on a linear programming problem and a quadraticallly constrained quadratic programming problems.

Total points: 30. Exercises are graded by completion, assignments are graded by correctness.

I. Exercises from textbook chapter 4 (7 pts, 1pt for each problem)

4.1, 4.8, 4.11, 4.15, 4.21, 4.39, 4.47.

II. Assignments (23 pts)

II.1 Linear Programming: You are free to use any software packages. (10 pts)

Given

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & -1 & 2 \end{bmatrix},$$

$$b^T = \begin{bmatrix} -1 & 2 & 3 & -4 \end{bmatrix},$$

$$c^T = \begin{bmatrix} 1 & -2 & 1 & -1 \end{bmatrix},$$

and $n = 4$, solve the following linear programming problems. If a solution is found, validate that the solution satisfies the optimality criteria (which was talked about in class or textbook). Otherwise, explain why a solution is not feasible and suggest how to mitigate the issue if you are the project leader:

II.1.1. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n$.

II.1.2. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n$.

II.1.3. minimize $f_0(x) = c^T x$ subject to $Ax \leq b$, $x \in R^n_+$.

II.1.4. minimize $f_0(x) = c^T x$ subject to $Ax = b$, $x \in R^n_+$.

Solutions

II.1.1. Let $z = [-3 \ -1 \ 0 \ 0]^T$. Let $x = tz$, with $t \in [4/9, \infty)$. Check if all the inequalities are satisfied:

$$\begin{aligned} x_1 + x_4 = -3t + 0 &\leq -1 \\ x_2 = -t &\leq 2 \\ x_3 = 0 &\leq 3 \\ 2x_1 + 3x_2 - x_3 + 2x_4 = -9t &\leq -4 \end{aligned}$$

Clearly, as $t \to \infty$, all these inequalities will remain satisfied. Since the feasible set isn't empty, this problem isn't infeasible. Then, $c^T x = 1 \cdot (-3t) + (-2) \cdot (-t) + 0 + 0 = -t$, which means $\lim_{t\to\infty} c^T x = \lim_{t\to\infty} tc^T z = \lim_{t\to\infty}(-t) \to -\infty$. Since the objective is to minimize here, this problem is unbounded. To alleviate this, we can constrain the problem further, something like $x_1, x_2, x_3 \geq 0$ and $x_4 \leq 0$ leads to a bounded solution.

II.1.2.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 2 & 3 & -1 & 2 & -4 \end{array}\right] \xrightarrow{R_4 \leftarrow R_4 - 2R_1 - 3R_2 + R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right] \implies 0 = -5$$

Since $b \notin \text{range}(A)$, we have a contradiction. This means that the feasible set is empty, and the solution is infeasible. What we can do here is just relax some of the equality constraints into inequality constraints, particularly $x_2 \leq 2$ and $2x_1 + 3x_2 - x_3 + 2x_4 \leq -4$. This makes the problem unbounded, at which point we can add some more inequality constraints $x_3 \geq 0$ and $x_4 \leq 0$ to make the problem bounded.

II.1.3. For $x \in \mathbb{R}^n_+$, the inequality $x_1 + x_4 \leq -1$ cannot be achieved. This means that the feasible set is empty, and the solution is infeasible. We can sometimes give up on the nonnegativity constraints in these circumstances, which we can bound with the techniques explained in (II.1.1). There may be situations where we must have a nonnegative x, at that point we can choose to throw away or alter some inequality constraints to relax the problem. Particularly, getting rid of $x_1 + x_4 \leq -1$ and $x_3 \leq 3$ allows the problem to be both feasible and bounded.

II.1.4 For the same reasoning as (II.1.2), the feasible set is empty, and the solution is infeasible. Since we also need to deal with nonnegativity here, a simple solution would be to throw away some equality constraints directly. Particularly, getting rid of $x_1 + x_4 = -1$ and $x_3 = 3$ allows the problem to be both feasible and bounded.

II.2 Graph embedding (13 pts) Graph embedding is an important problem in machine learning and graph theory. Given an undirected graph $G = (V, E)$ with n vertices, the problem is to assign coordinates in $\mathbb{R}^m$ to each vertex $v \in V$. Typically there are desired qualities or constraints imposed on the embedding—e.g. the coordinates assigned to connected nodes should be close with respect to some distance metric. We can formulate this as a quadratically constrained quadratic program (QCQP). Let $A \in \{0,1\}^{n \times n}$ be the symmetric adjacency of G, and let D be the corresponding diagonal degree matrix such that $D_{ii} = \sum_j A_{i,j}$. The *graph Laplacian* is defined to be $L = D - A$.

Let $x, y \in \mathbb{R}^n$ represnt the x and y coordinates of n vertices. Given a parameter $c \in \mathbb{R}_{++}$, one way to define the graph embedding problem in 2-d is to solve the following problem:

$$\begin{aligned} \min_{x,y \in \mathbb{R}^n} \; & x^\top L x + y^\top L y \\ \text{s.t.} \quad & x^\top x \leq c, \quad y^\top y \leq c \end{aligned} \tag{1}$$

(i) Show that $x^\top L x = \sum_{i,j \in E} (x_i - x_j)^2$

(ii) Consider a partitioning of x; $x = [x_1 : x_2]^\top$, where $x_1 \in \mathbb{R}^{n-k}$ corresponds to the coordinates of $n - k$ "free" nodes and $x_2 \in \mathbb{R}^k$ are the coordinates of k "fixed"/"anchor" nodes (likewise for y). Under these "fixed-node" constraints, show that Prob. 1 is equivalent to

$$\begin{aligned} \min_{x_1, y_1 \in \mathbb{R}^{n-k}} \; & x_1^\top L' x_1 + y_1^\top L' y_1 + b^\top x_1 + d^\top y_1 \\ \text{s.t.} \quad & x_1^\top x_1 \leq c'_x, \quad y_1^\top y_1 \leq c'_y \end{aligned}$$

In other words, express L', b, d, and c'_x & c'_y in terms of x_1, x_2, y_1, y_2, L, and c. Are there any issues with Prob. 1 if there are no fixed nodes?

(iii) Implement the problem in CVX/CVXPY and show your result for the given graph with $c'_x = c'_y = 10$.

(iii.a) We have written a partial framework in Python to get you started:
`https://colab.research.google.com/drive/1apgxNJGN1E4_W6awYbbhNxTyL0VvvMVH?usp=sharing`.

(iii.b) If you prefer a different language, you can also download a .txt file containing L, x, y, and the indices of the fixed nodes:

https://piazza.com/class_profile/get_resource/kx85xrdgigl5m5/kzfw6ud6fd964c (idx, x, y are the first 3 columns)
(iv) Suppose we change the quadratic inequality constraints on x and y to equality constraints and add a constraint $x^\top y = c'_{xy}$. Is the problem still convex? If not, can we still recover a solution?

Solutions

(i)

$$\begin{aligned}
x^\top Lx &= x^\top (D-A)x = x^\top Dx - x^\top Ax \\
&= \sum_i D_{ii}x_i^2 - \sum_{i,j\in E} 2x_i x_j \\
&= \sum_i \sum_{i,j\in E} x_i^2 - \sum_{i,j\in E} 2x_i x_j \\
&= \sum_{i,j\in E} (x_j^2 + x_i^2 - 2x_i x_j) = \sum_{i,j\in E} (x_i - x_j)^2
\end{aligned}$$

(ii)

1-d case:

consider a partitioning of L that is implied by our partitioning of x:

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

Now, expanding the quadratic:

$$\begin{aligned}
x^\top Lx &= [x_1 : x_2] L [x_1 : x_2]^\top \\
&= [x_1 : x_2] \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} [x_1 : x_2]^\top \\
&= x_1^\top L_{11} x_1 + 2x_2^\top L_{21} x_1 + x_2^\top L_{22} x_2
\end{aligned}$$

So, $b = 2x_2^\top L_{21}$ and $c'_x = c - x_2^\top x_2$. Additionally, note that the minimizer of $x_1^\top Lx_1 + x_1^\top b$ is equivalent to the minimizer of $x_1^\top Lx_1 + x_1^\top b + x_2^\top L_{22} x_2$

(iii)

https://colab.research.google.com/drive/1VZZNL8PTxnLuzaDtX_7g7kAb8q4fNJt4?usp=sharing

(iv)

Changing the quadratic inequality constraints to inequality constraints results in a *non-convex* problem. With inequality constraints, the feasible set is the intersection of convex *ball* constraints (i.e. convex). With equality constraints, the feasible set is the intersection of non-convex sphere constraints (e.g. a circle in 2-d).

However, an optimal solution can still be recovered. Noting that L is real & symmetric and writing out the dual problem to P1 results the unique solution to the dual variables corresponding to the eigenvalues of L, and the primal variables are the corresponding eigenvalues. Alternatively, the solution to P1 is equivalent to the solution of a Rayleigh Quotient minimization problem so we can use the Variational Theorem of Eigenvalues.