

CSE 203B W21 Homework 1

In this homework, we work on exercises from text book including Voronoi diagram (2.7, 2.9), quadratic function (2.10), general sets (2.12), cones and dual cones (2.28, 2.31, 2.32). Extra assignments are given on convex sets.

Total points: 27. Exercises are graded by completion, assignments are graded by content.

I. Exercises from textbook chapter 2 (7 pts, 1pt for each problem)

2.7, 2.9, 2.10, 2.12, 2.28, 2.31, 2.32.

II. Assignments (20 pts)

II. 1. True or False with one-sentence explanation. (6 pts)

(1) The union of two convex sets is always non-convex.

Ans: False.

(2) Set $\{(x, y) | x^2 y < 3, x, y \in R\}$ is convex.

Ans: False.

(3) Set $\{(x^2 - y, y^3) | x + 4y < 3, x, y \in R\}$ is convex.

Ans: False.

(4) For every matrix $A \in R^{m \times n}$ where $m < n$, the null space of matrix A is convex.

Ans: True.

(5) For every matrix $A \in R^{m \times n}$ where rank A is m , the range of matrix A is convex.

Ans: True.

(6) Given a matrix $X \in R^{n \times n}$ such that $y^T X y > 0$ for all $y \in R^n$, then matrix X has to be symmetric, i.e. $X = X^T$.

Ans: False.

II. 2. Support vector machine (SVM): Given two sets of points $C = \{x_1, \dots, x_m\}$ and $D = \{y_1, \dots, y_m\}$, where $x_i, y_i \in R^n$, we find a hyperplane with vector $a \in R^n$ and bias $b \in R$ to maximize the following objective function.

$$\max u \in R \tag{1}$$

$$\text{s.t. } a^T x_i \leq b - u, \quad a^T y_i \geq b + u, \quad i = 1, \dots, m \tag{2}$$

$$\|a\|_2 \leq 1 \tag{3}$$

Note that in the above expressions, the variables are vector a , and scalars b and u .

- (1) Rewrite equation 2 as two simultaneous linear inequalities in matrix format. (2 pts)

Ans:

$$\begin{bmatrix} x_1^T \\ x_2^T \\ \dots \\ x_m^T \end{bmatrix} a \leq \begin{bmatrix} b-u \\ b-u \\ \dots \\ b-u \end{bmatrix}, \begin{bmatrix} y_1^T \\ y_2^T \\ \dots \\ y_m^T \end{bmatrix} a \geq \begin{bmatrix} b+u \\ b+u \\ \dots \\ b+u \end{bmatrix} \quad (4)$$

- (2) Use explicit form to express the two convex hulls of sets C and D . (2 pts)

Ans:

$$\mathbf{conv} C : \left\{ \sum_{i=1}^m \theta_i x_i \mid x_i \in C, \sum_{i=1}^m \theta_i = 1, \theta_i \geq 0 \forall i \right\} \quad (5)$$

$$\mathbf{conv} D : \left\{ \sum_{i=1}^m \theta_i y_i \mid y_i \in D, \sum_{i=1}^m \theta_i = 1, \theta_i \geq 0 \forall i \right\} \quad (6)$$

- (3) Create a numerical example with $m = 5, n = 2$. (2 pts)

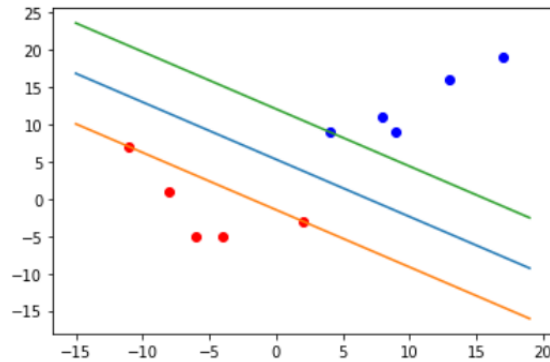
Ans:

$$C = \begin{bmatrix} 13 & 16 \\ 17 & 19 \\ 4 & 9 \\ 9 & 9 \\ 8 & 11 \end{bmatrix}, D = \begin{bmatrix} -8 & 1 \\ -4 & -5 \\ 2 & -3 \\ -6 & -5 \\ -11 & 7 \end{bmatrix} \quad (7)$$

- (4) Use a nonlinear programming package (e.g. Matlab) to derive the solution of item (3). (2 pts)

Ans:

$$a = [-0.61, -0.79], b = [-4.21], u = [5.37] \quad (8)$$



II. 3. Given p hyperplanes

$$a_i x = b_i, \text{ for } i = 1, 2, \dots, p, \quad x \in \mathbb{R}^n.$$

List the maximum number of disjoint regions separated by the hyperplanes for the following cases. (6 pts)

(1) $n = 2, p = 3$. **Ans:** 7

(2) $n = 2, p = 6$. **Ans:** 22

(3) $n = 3, p = 3$. **Ans:** 8

(4) $n = 3, p = 7$. **Ans:** 64

Generalize the problem to any given n and p and write down the equation.

For example, $N(n, p) = 1 + p$ if $n = 1$.

Ans: The recurrence relation is

$$N(n, p) = \binom{p}{0} + \binom{p}{1} + \binom{p}{2} + \dots + \binom{p}{n}.$$