

## CSE 203B W21 Homework 0

Due Time : 11:50pm, Monday Jan. 18, 2021 Submit to Gradescope  
Gradescope: <https://gradescope.com/>

In this homework, we work on the basics of linear algebra.

Total points: 34. All the problems are graded by content.

### 1. Matrix Properties (14 pts)

#### 1.1. Linear System (2pts)

Consider the following system of equations

$$\begin{aligned}2x_1 + 4x_2 + 6x_3 &= 1 \\x_1 - x_2 + 2x_3 &= -1 \\3x_1 + 5x_3 &= 2\end{aligned}$$

Write the equations in a matrix form.

1.2. For the matrix in problem 1.1, calculate its range. What's the rank of this matrix? (2pts)

1.3. Calculate the nullspace of the matrix in problem 1.1. What's the relation between the range and nullspace of a matrix? (2pts)

1.4. Calculate the trace and determinant of the matrix in problem 1.1. Find the eigenvalues and eigenvectors. (2pts)

1.5. Prove the following properties. (3 pts)

- For  $A \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $\text{tr}AB = \text{tr}BA$ .
- For  $A, B \in \mathbb{R}^{n \times n}$ ,  $\det AB = \det A \det B$ .
- For  $A \in \mathbb{R}^{n \times n}$ ,  $\det A = \prod_{i=1}^n \lambda_i$  where  $\lambda_i, i = 1, \dots, n$  are the eigenvalues of  $A$ .

1.6. Provide a simple but nontrivial numerical example to demonstrate the three equations in problem 1.5. Let us use matrix  $A \in \mathbb{R}^{3 \times 3}$ ,  $\text{tr}A = 5$ , and matrix  $B \in \mathbb{R}^{3 \times 3}$ ,  $\text{tr}B = 4$ . (3 pts)

### 2. Matrix Operations (20 pts)

**Gradient:** consider a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  that takes a vector  $x \in \mathbb{R}^n$  and returns a real value. Then the gradient of  $f$  (w.r.t.  $x$ ) is the vector of partial derivatives, defined as

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}.$$

**Hessian:** consider a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  that takes a vector  $x \in \mathbb{R}^n$  and returns a real value. Then the Hessian matrix of  $f$  (w.r.t.  $x$ ) is the  $n \times n$  matrix of partial derivatives, defined as

$$\nabla_x^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}.$$

2.1. Derive the gradient and Hessian matrix for the linear function

$$f(x) = b^T x$$

where  $x \in \mathbb{R}^n$  and vector  $b \in \mathbb{R}^n$ . (2 pts)

2.2. Derive the gradient and Hessian matrix of the quadratic function (4 pts)

$$f(x) = x^T A x + b^T x + c$$

where  $x \in \mathbb{R}^n$ , matrix  $A \in \mathbb{S}^n$  is symmetric, and vectors  $b, c \in \mathbb{R}^n$ . (2 pts)

2.3. Derive the gradient of the function  $f : \mathbb{S}^n \rightarrow \mathbb{R}$

$$f(X) = \log \det X, \quad \text{dom} f = \mathbb{S}_{++}^N. \quad (4 \text{ pts})$$

2.4. Provide a simple but nontrivial numerical example to demonstrate the gradient and Hessian matrix in problem 2.2 and gradient in problem 2.3. (4 pts)

2.5. Given matrix  $A \in \mathbb{R}^{m \times n}$  where  $m > n$  and  $\text{rank}(A) = n$ , and vector  $b \in \mathbb{R}^m$  where  $b \notin \mathcal{R}(A)$ , find vector  $x \in \mathbb{R}^n$  such that vector  $Ax$  is as close as possible to vector  $b$ , measured by the square of the Euclidean norm  $\|Ax - b\|_2^2$ . (4 pts)