CSE 203B W21 Homework 3

Due Time : 11:50pm, Monday Feb. 8, 2021 Submit to Gradescope

In this homework, we work on exercies from text book. Problem 4.1, 4.8, 4.12 and 4.15 are related to LP. Problem 4.21 and 4.39 are related to QCQP and SDP, respectively. Also, we practice using the convex optimization tools on a compressed sensing problem and a semi-definite programming problem.

Total points: 26. Exercises are graded by completion, assignments are graded by correctness.

I. Exercises from textbook chapter 4 (6 pts, 1pt for each problem)

4.1, 4.8, 4.12, 4.15, 4.21, 4.39.

II. Assignments (20 pts)

II. 1. Compressed Sensing (12 pts)

You are given the data (hw3_signal.txt), which is generated by a sparse linear combination of sinusoidal waves ($y(t) = \sum_k a_k sin(2\pi f_k t)$), and your job is to figure out the frequency components and amplitudes that compose the data. It is known that the data is a 5 second data sampled at 1k Hz. Also, the frequencies (in the unit of Hertz) are in the set of $\{i | i \in \mathbb{N}, 1 \leq i \leq 100\}$.

Please formulate the problem into an optimization problem that minimizes the sum of the L_2 norm of the noise (at the sensor) and the L_1 norm of the frequency components. Once you derive the formulation, use the convex optimization programming tool to compute the numerical values of the amplitudes and figure out the frequencies of the signal.

(1) Write the formulation. Note that we need a weight to balance between the norms of the noise and the frequency components in the objective function.

$$min \quad \alpha ||x||_1 + ||Ax - y||_2^2$$

(2) Show your results with three or more samples of weights in (1).

The frequency components are 2Hz, 10Hz, and 79Hz, and the magnitudes are 2.0, 1.0 and 0.5, respectively.

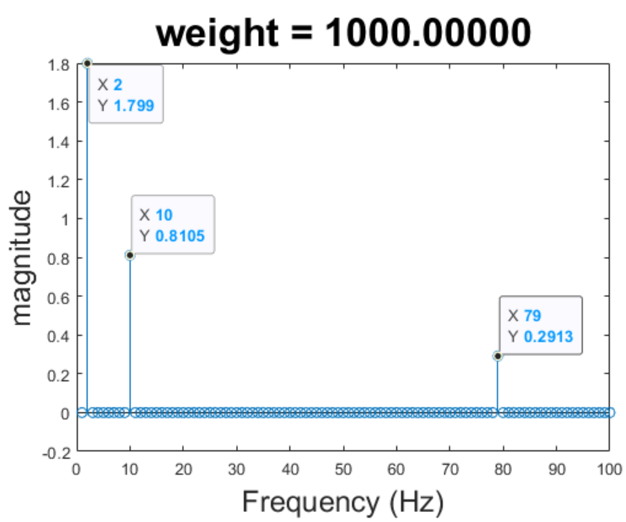
(3) Show and explain your best choice of the weight.

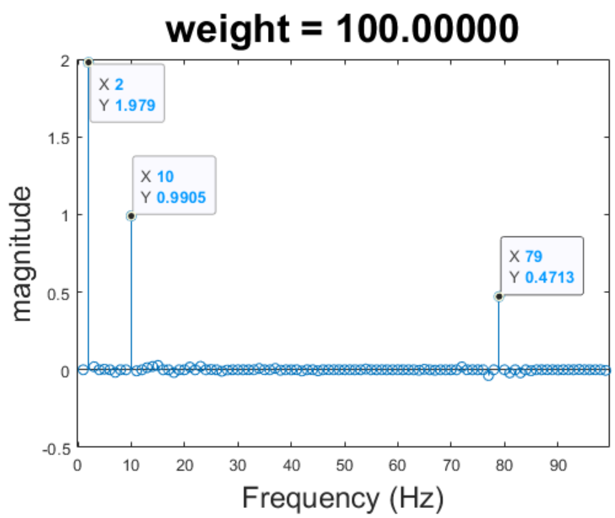
For larger α, the sparsity is enforced and the noise term shrinks. Through experimenting with different weights, we observe the magnitudes converge around $\alpha < 0.1$.

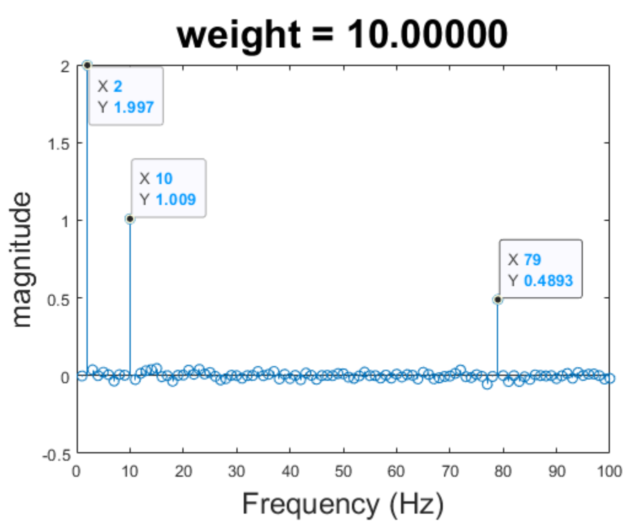
Hint: You should get the major three frequency components in your solution.

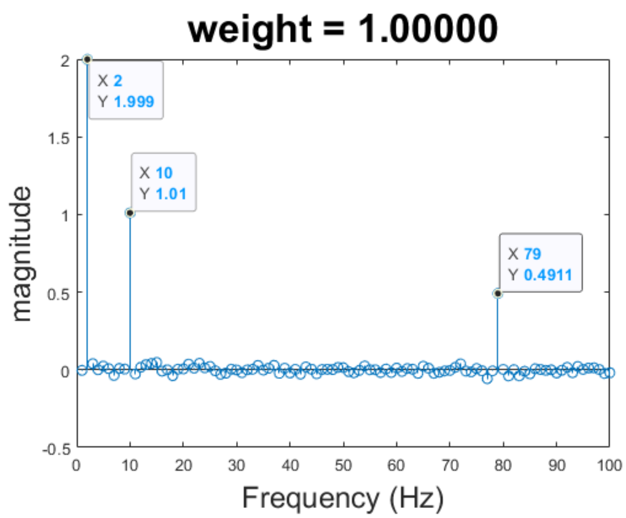
```
t = 0:0.001:5;
A = zeros(length(t) , 1e2 );

for k=1:1e2
   A(:,k) = sin(2*pi*k*t);
end

alpha = 1e-1;

% cvxopt

cvx_clear;

n = 1e2;

cvx_begin
    variable x(n)
    minimize( alpha*norm( x , 1 ) + (A*x-y)' * (A*x-y)   )
cvx_end
```

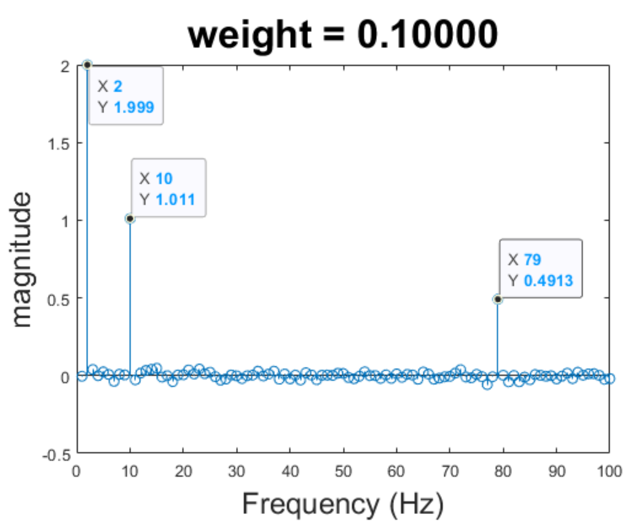
Figure 1: CVX codes for assignment 1

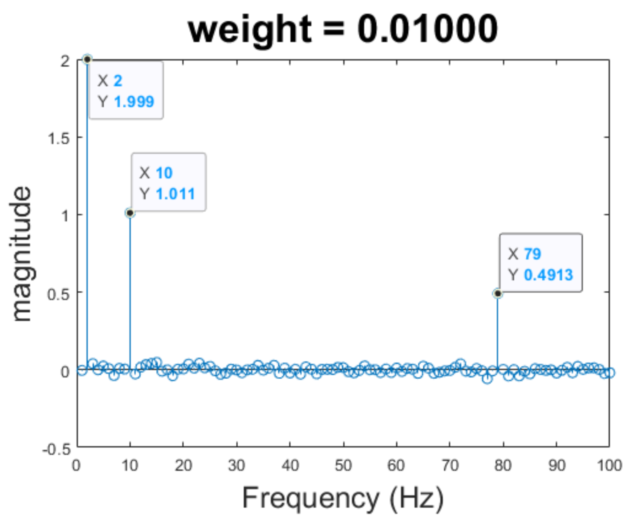


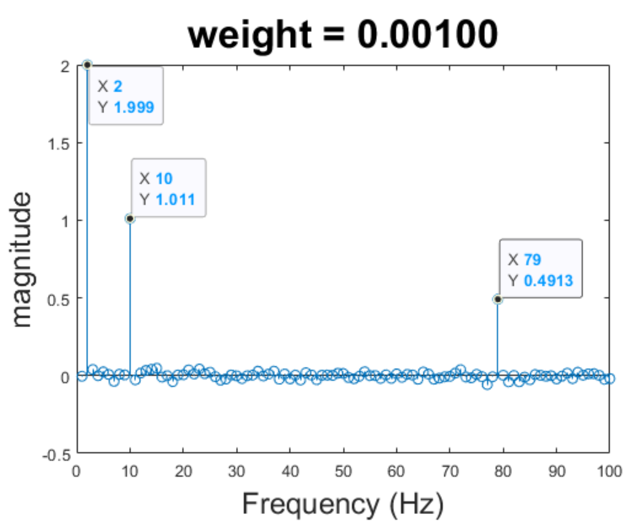

weight = 100.00000
X 2
Y 1.979
X 10
Y 0.9905
X 79
Y 0.4713
magnitude
Frequency (Hz)


weight = 10.00000
X 2
Y 1.997
X 10
Y 1.009
X 79
Y 0.4893
magnitude
Frequency (Hz)


weight = 1.00000
magnitude
Frequency (Hz)
X 2
Y 1.999
X 10
Y 1.01
X 79
Y 0.4911
2
1.5
1
0.5
0
-0.5
0
10
20
30
40
50
60
70
80
90
100


weight = 0.10000
magnitude
Frequency (Hz)
X 2
Y 1.999
X 10
Y 1.011
X 79
Y 0.4913
2
1.5
1
0.5
0
-0.5
0
10
20
30
40
50
60
70
80
90
100


weight = 0.01000
X 2
Y 1.999
X 10
Y 1.011
X 79
Y 0.4913
magnitude
Frequency (Hz)


weight = 0.00100
X 2
Y 1.999
X 10
Y 1.011
X 79
Y 0.4913
magnitude
Frequency (Hz)

ps: CVX is a Matlab-based modeling system for convex optimization created by Professor Stephen Boyd. For more details, please refer to the tutorial for CVX: `https://web.stanford.edu/class/ee364a/lectures/cvx_tutorial.pdf`

II. 2. Semi-definite Programming (8 pts)

Given a matrix $R \succ 0$ and a point z we can define an ellipsoid in R^n: $E_{R,z} := \{y|(y-z)^\top R(y-z) \leq 1\}$. One can prove that the volume of $E_{R,z}$ is proportional to $\sqrt{det(R^{-1})}$. Suppose we are given a convex set $X \in R^n$ described as the convex hull of k points $c_1, \ldots, c_k$. We would like to find an ellipsoid circumscribing these k points that has minimum volume.

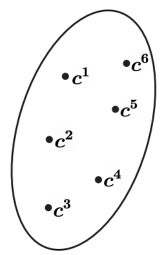


Figure 2: Illustration of the circumscribed ellipsoid problem

This problem can be written in the following form:

$$min_{R,z} \; vol(E_{R,z})$$

$$s.t. \; c_i \in E_{R,z}, i = 1, \ldots, k$$

,

which is also equivalent to:

$$min_{R,z} \quad -\log detR$$

$$s.t. \; (c_i - z)^\top R(c_i - z) \leq 1, i = 1, \ldots, k$$

,

$$R \succ 0$$

.

(1) Convert this problem to SDP.

Let $R = M^T M$, we have

$$
\begin{aligned}
min_{M,z} \quad & -2\log det M \\
s.t. \ (c_i - z)^\top M^\top M (c_i - z) & \leq 1, i = 1, \dots, k \\
M & \succ 0
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
min_{M,z} \quad & -2\log det M \\
s.t. \begin{bmatrix} I & Mc_i - Mz \\ (Mc_i - Mz)^\top & 1 \end{bmatrix} & \succeq 0, i = 1, \dots, k \\
M & \succ 0
\end{aligned}
$$

let $y = Mz$, we will have

$$
\begin{aligned}
min_{M,y} \quad & -2\log det M \\
s.t. \begin{bmatrix} I & Mc_i - y \\ (Mc_i - y)^\top & 1 \end{bmatrix} & \succeq 0, i = 1, \dots, k \\
M & \succ 0
\end{aligned}
$$

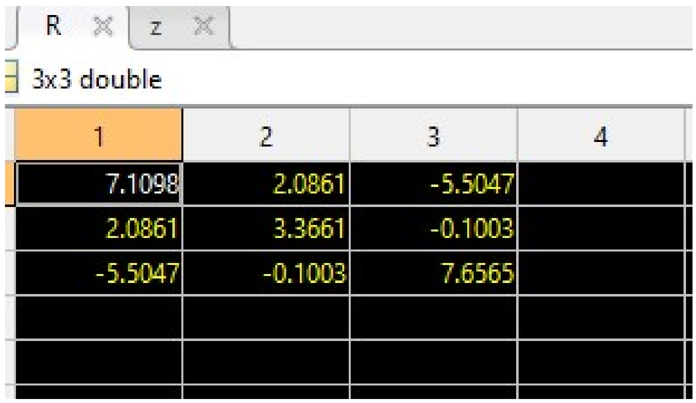
```
x = [ 1.4, 0.8, 1.3; 0.4, 1.0, 0.7 ; 0.9, 0.3, 0.6; 0.7, 1.2, 0.4; 1.4, 0.8, 1.1; 1.5, 0.1, 1.3 ];
x = x';

% cvxopt

cvx_clear;
n = 3;
m = size(x,2);
cvx_begin
    variable M(n,n) semidefinite;
    variable y(n);
    minimize( -2 * log_det(M) );
    subject to
        norms( M * x - y * ones(1,m) , 2 ) <= 1;

cvx_end

%%
R = M' * M;
z = inv(M) * y;
```

Figure 3: CVX codes for assignment 2

(2) Given 6 points in 3D space:

$\{(1.4, 0.8, 1.3), (0.4, 1.0, 0.7), (0.9, 0.3, 0.6), (0.7, 1.2, 0.4), (1.4, 0.8, 1.1), (1.5, 0.1, 1.3)\}$.

Find R, z for the minimum volume ellipsoid $E_{R,z} := \{y|(y-z)^\top R(y-z) \le 1\}$.
(You are allowed to use any convex optimization tools).



1	2	3	4
7.1098	2.0861	-5.5047	
2.0861	3.3661	-0.1003	
-5.5047	-0.1003	7.6565	

Figure 4: Solution of R

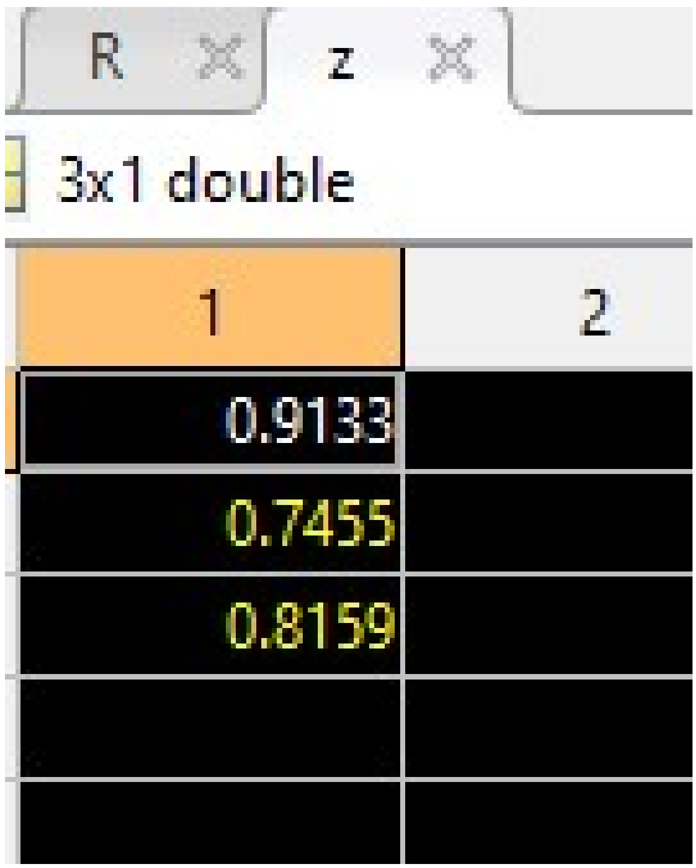


Figure 5: Solution of z