CSE203B Convex Optimization Midterm Examination, 2/18/2020 Name _____________

1 True or False

Circle your choice of true or false. Use a short sentence to explain your choice. (20 points)

1. The union of two convex sets is convex.
   True False
   False. Union operation does not preserve convexity.

2. Given two convex sets S_1 and S_2 in the same domain, then set $S_3 = \{x | x \in S_1, x \notin S_2\}$ is convex.
   True False
   False. Similar reason as (1).

3. Given two convex sets $S_1, S_2 \subset R^n$, then set $S_3 = \{x_1 - x_2 |\ x_1 \in S_1,\ x_2 \in S_2\}$ is also convex.
   True False
   True. The sum of two convex sets is still convex.

4. Function $f(x) = -xlogx,\ x \in R_{++}$ is a convex function
   True False
   False. Concave.

5. The conjugate function $f^*(y)$ is convex even if function $f(x)$ is not convex.
   True False
   True. It is the pointwise maximum of a family of convex functions of y.

6. Function $f(x) = log \sum_{i=1:n} e^{a_i \times x_i}$, where a_i and $x_i \in R$ for $i = 1, ..., n$, is convex.
   True False
   True. Refer to homework 2.

7. The equation $x_1^3 x_2^{-1} + x_3^2 x_4^5 = 0$ for $x \in R_+^4$ can be converted into a linear equality constraint using the standard geometric programming formulation.
   True False
   False. Not linear.

8. In geometric programming, a posynomial function $f(x)$ may not be convex, but can be converted into convex form.
   True False
   True. See geometric programming procedure in the textbook.

9. In second order cone programming, the set $\{x | ||Ax + b||_2 \leq c^T x + d\}$, where $A \in R^{m \times n}$, $x, b, c \in R^n$ and $d \in R$, is convex.
   True False
   True/False. The second order norm is a convex set./There are typos in dimensions.

10. The inequality $sup_{z \in Z} inf_{w \in W} f(w,z) \leq inf_{w \in W} sup_{z \in Z} f(w,z)$ is true even when function $f(w,z)$ is not convex.
    True False
    True. It is the statement of max-min property.

2 Theorems and Proofs

Problem 2.1 Prove the following optimality criterion for a convex optimization problem. Suppose that the problem is convex and the objective function $f_0(x)$ is differentialble, prove that $\bar{x}$ is optimal if and only if $\bar{x}$ is feasible and $\nabla f_0(\bar{x})^T(y-\bar{x}) \geq 0$ for all feasible y. (10 points)

Given $f_0(x)$ is convex and differentiable, show that $\bar{x}$ is optimal $\iff$ $\bar{x}$ is feasible and $\bigtriangledown f_o(\bar{x})^T(y-\bar{x}) \geq 0$ for all feasible y.

$\Leftarrow$: Since f_o is convex over x and differentiable, the first order condition holds. Given $\bigtriangledown f_o(\bar{x})^T(y-\bar{x}) \geq 0$, we have $f_o(y) - f_o(x)$ for all feasible y, which shows that $\bar{x}$ is optimal.

$\Rightarrow$: Suppose $\bigtriangledown f_o(\bar{x})^T(y-\bar{x}) < 0$ for all y, and let $z = \lambda y + (1-\lambda)\bar{x}$, then following the first order condition, we have $f_o(z) = f_o(x) + \bigtriangledown f_o(\bar{x})^T(z-\bar{x})$. since $\bigtriangledown f_o(\bar{x})^T(z-\bar{x}) < 0$, we arrive at $f_o(z) < f_o(x)$, which contradicts to the statement that $\bar{x}$ is optimal. Therefore, we show that given $\bar{x}$ is optimal, $\bigtriangledown f_o(\bar{x})^T(y-\bar{x}) \geq 0$ for all y.

Problem 2.2 Prove that the Lagrange dual function is concave even if the primal problem is not convex. (10 points)

$$
\begin{aligned}
g(\lambda, \nu) &= inf_{x \in D} L(x, \lambda, \nu) \\
&= inf_{x \in D}(f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x))
\end{aligned} \tag{1}
$$

Since the dual function is the pointwise infimum of a family of affine functions of (λ, ν), it is concave even when the primal problem is not convex.

3 Case Studies

Problem 3.1 Dual Cone: Given a cone $K = \{\theta_1 u_1 + \theta_2 u_2 \mid u_1 = [2, -1, 3]^T,\ u_2 = [-2, 1, 0]^T,\ \theta_1 \geq 0, \theta_2 \geq 0\}$, find the dual cone of K. (15 points)

Ans: The cone can be expressed as

$$K = \{A\theta \mid \theta \succeq 0\}, \quad A = [u_1, u_2]$$

According to the definition of dual cone, we have

$$K^* = \{x \mid A^T x \succeq 0\} = \{x \mid \begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 0 \end{bmatrix} x \succeq 0\}$$

For the explicit expression, you need to find a basis for the dual cone. Notice that $A^T x \succeq 0$ expresses the intersection of two halfspaces $u_1^T x \geq 0$ and $u_2^T x \geq 0$, as shown in the following figure.

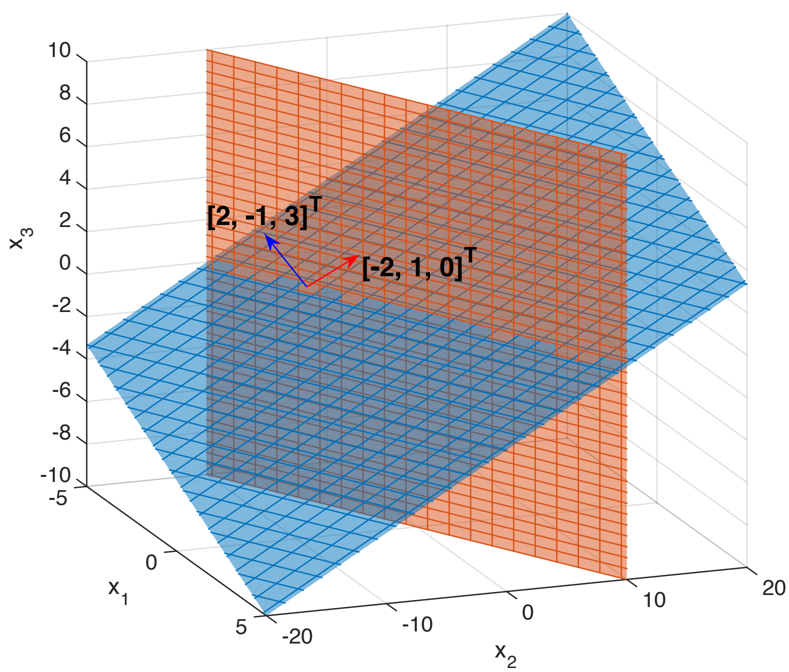

Figure 1: The dual cone is expressed as the intersection of two halfspaces $u_1^T x \geq 0$ and $u_2^T x \geq 0$.

To explicitly express the dual cone, we need four vectors. For example, we can choose

$$K^* = \{\theta_1 u_1 + \theta_2 u_2 + \theta_3 u_3 + \theta_4 u_4 \mid u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ u_2 = \begin{bmatrix} -2 \\ 1 \\ 5/3 \end{bmatrix},\ u_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ u_4 = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix},\ \theta_i \geq 0 \text{ for } i = 1, \ldots, 4\}.$$

Problem 3.2 Conjugate Function: Given a function $f(x) = x_1 + 2x_2 + 3x_3^2,\ x \in R^3$, derive the conjugate function $f^*(y), y \in R^3$. (15 points)

Ans: According to the definition, the conjugate function is expressed as

$$f^*(y) = \sup_{x \in \mathrm{dom}\, f} (y^T x - f(x)).$$

We have the gradient of function

$$\nabla_x (y^T x - f(x)) = y - \begin{bmatrix} 1 \\ 2 \\ 6x_3 \end{bmatrix}.$$

If $y_1 \neq 1$ or $y_2 \neq 2$, the function is unbounded. Therefore, we have the domain $y_1 = 1$ and $y_2 = 2$. When $y_3 = 6x_3$ the supremum is achieved.

$$f^*(y) = \begin{cases} \frac{y_3^2}{12},\ y_1 = 1,\ y_2 = 2 \\ \infty,\ otherwise \end{cases}.$$

Problem 3.3 Primal Dual Formulation: Given a linear programming problem,

minimize $f_0(x) = c^T x$

subject to $Ax \leq b$, and $Px = q$, where $x \in R^n_+$ (i.e. $x \succeq 0$).

Derive the dual problem. (10 points)

The Lagrangian:

$$
\begin{aligned}
L(x,\lambda,\nu) &= c^T x + \lambda^T(Ax-b) + \nu^T(Px-q) + \lambda_2^T(-x) \\
&= (c^T + \lambda^T A + \nu^T P - \lambda_2^T)x - \lambda^T b - \nu^T q
\end{aligned}
$$

The dual function:

$$g(\lambda,\nu) = inf_x \, L(x,\lambda,\nu)$$

$$g(\lambda,\nu) = \begin{cases} -\lambda^T b - \nu^T q & c + A^T\lambda + P^T\nu - \lambda_2 = 0 \\ -\infty & Otherwise \end{cases}$$

As $\lambda_2 \succeq 0$, $c + A^T\lambda + P^T\nu - \lambda_2 = 0$ is equivalent to $c + A^T\lambda + P^T\nu \succeq 0$

So the dual problem is:

$$
\begin{gathered}
\max -\lambda^T b - \nu^T q \\
\text{s.t. } \; c + A^T\lambda + P^T\nu \succeq 0 \\
\lambda \succeq 0
\end{gathered}
$$

4 Problems from Exercises

Problem 4.1 Let $C \subset R^n$ be the solution set of a quadratic inequality, $C = \{x \in R^n | x^T Ax + b^T x + c \leq 0\}$, with $A \in S^n, b \in R^n$, and $c \in R$.
Prove that C is convex if $A \succeq 0$. (10 points)

Ans: You can proof this by the definition of convex sets, or by proving $f(x) = x^\top Ax + b^\top + c$ is a convex function.

Since $\nabla f(x) = (A + A^\top)x$, $\nabla^2 f(x) = A + A^\top = 2A$ (as A is symmetric).

If $A \succeq 0$, then the hessian $\nabla^2 f \succeq 0$, which means $f(x)$ is a convex function.

Any level set for a convex function is a convex set. So C is a convex set.

(You can also prove this by examine the intersection of C and arbitrary lines)

Problem 4.2 Prove that the following function is convex. (10 points)

$f(x) = 1/(x_1 x_2)$, where $x \in R^2_{++}$.

Ans: Calculate the correct Hessian and examine all the principle minors to prove the Hessian is PSD. You can use other methods to prove the Hessian is PSD, e.g. factorization; for arbitrary x, $x^\top Hx \geq 0$.

The Hessian is:

$$H(f) = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}$$

As $x \in R^2_{++}$, $\frac{2}{x_1^3 x_2} > 0$, $\frac{2}{x_1 x_2^3} > 0$, the determinate $\frac{2}{x_1^3 x_2} \frac{2}{x_1^3 x_2} - \frac{1}{x_1^2 x_2^2} \frac{1}{x_1^2 x_2^2} > 0$. So the Hessian is PSD, the function is a convex function.