CSE 203B W21 Midterm 10AM 2/16/2021-10AM 2/18/2021
Submit your solution to gradescope before the due time.

Policy of the Exam: Here is the policy of the exam:
1. This is an open-book take-home exam. Internet search is permitted. However, you are required to work by yourself. Consultation or discussion with any other parties is not allowed.
2. You are not required to typeset your solutions. We do expect your writing to be legible and your final answers clearly indicated. Also, please allow sufficient time to upload your solutions.
3. You are allowed to check your answers with programs in Matlab, CVX, Mathematica, Maple, NumPy, etc. Be aware that these programs may not produce the intermediate steps needed to receive credit.
4. If something is unclear, state the assumptions that seem most natural to you and proceed under those assumptions. Out of fairness, we will not be answering questions about the technical content of the exam on Piazza or by email. The solution will then be graded based on the reasonable assumptions made.

Part I: True or False: Explain your answer with one sentence (36 pts)

Problem 1 (convex set): Set $\{(x^3, y^2 - x)|x + 4y < 3, x, y \in R\}$ is convex.
T/F:

Problem 2 (dual cone): Given cone
$K = \{\theta_1 u_1 + \theta_2 u_2 | u_1 = [2, -1]^T, u_2 = [1, 0]^T, \theta_1 \geq 0, \theta_2 \geq 0\}$, its dual cone is
$K^* = \{x_1 u_1 + x_2 u_2 | u_1 = [1, 2]^T, u_2 = [0, -1]^T, x_1 \geq 0, x_2 \geq 0\}$.
T/F:

Problem 3 (Convex Function): Given function $f(x, y) = x^T A x + 2x^T B y + y^T C y$, where matrices $A, C \in S^n$ and $x, y \in R^n$, then $f(x, y)$ is concave if and only if the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ is negative semidefinite.
T/F:

Problem 4 (Convex Function): Function $h(x) = \sum_{i=1}^{m} a_i f_i(x)$ is convex if function $f_i(x)$ is convex and $a_i \in R$ for every $i \in \{1, ..., m\}$.
T/F:

Problem 5 (Convex Function): Function $h(x) = f(x) \times g(x)$ is convex if both functions $f(x)$ and $g(x)$ are convex.
T/F:

Problem 6 (Quadratic Optimization Problem): A second-order cone programming (SOCP) problem is solved as a typical quadratically constrained quadratic programming (QCQP) problem.

T/F:

Problem 7 (Geometric Programming): A posynomial function is a convex function.
T/F:

Problem 8 (Duality): Given a nonconvex programming problem as a primal problem, the **dual of its dual** has the same solution as the primal problem.
T/F:

Problem 9 (Duality): Given a function $f(x,y)$, the equality $\min_x \max_y f(x,y) = \max_y \min_x f(x,y)$ is always true.
T/F:

Part II: Problem Solving: Show your process

Problem 1. Support vector machine (SVM): Given two sets of points $C = \{x_1, ..., x_m\}$ and $D = \{y_1, ..., y_m\}$, where $x_i, y_i \in R^n$, we find a hyperplane with vector $a \in R^n$ and bias $b \in R$ to maximize the following objective function. (20 pts)

$$\max u, u \in R$$
$$\text{s.t. } a^T x_i \leq b - u, \quad a^T y_i \geq b + u, \quad for \ all \ i = 1, ..., m$$
$$||a||_2^2 \leq 1$$

(1) Given the above primal problem, formulate its dual problem.
(2) Create a nontrivial numerical example with $n = 2, m = 5$. Derive the solution via the primal formulation.
(3) Derive the solution of (2) via the dual formulation.
(4) Show the classification results of (2) and (3) with a 2-D plot (or plots).

Problem 2. Conjugate Function: Consider the function

$$f(x) = \begin{cases} ||x||_2^2, & ||x||_2 \leq a, \\ a(2||x||_2 - a), & ||x||_2 > a, \end{cases}$$

where variable $x \in R^n$ and constant $a \in R_{++}$. Derive the conjugate function $f^*(y)$, $y \in R^n$. (10 pts)

Problem 3. Kullback-Leibler Divergence: Show that $KL(p,q) = 0$ if and only if $p = q$. Here, we define the Kullback-Leibler (KL) divergence as $KL(p,q) = \sum_i p_i \log(\frac{p_i}{q_i})$, where $p_i > 0$, $q_i > 0$, $\forall i \in \{1, 2, ..., n\}$, and $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$. (10 pts)

Problem 4. Linear Programming Problem: Given the following optimization problem:

$$\text{minimize } c^T x$$
$$\text{subject to}$$
$$||x||_p = 1$$
$$-x \preceq 0$$

where variable $x \in R^n$, and constants $c \in R^n, p \geq 1$, derive an explicit solution.
(10 pts)

Problem 5. Compressed Sensing: Given the following problem

$$\text{minimize } ||y||_2^2 + \alpha ||x||_1$$
$$\text{subject to}$$
$$\mathbf{A}x + y = b$$

where variables $x, y \in R^n$, matrix $A \in R^{m \times n}$, and constant $b \in R^n$. (14 pts)
(1) Write the dual formulation assuming that x_i is positive for all $i \in \{1, ..n\}$.
(2) Repeat item (1) without the assumption that x_i is positive.