CSE203B Convex Optimization Midterm Examination, 2/14, Spring 2019 Name ______________________

1 True or False

Circle your choice of true or false. Use a short sentence to explain your choice. (20 points)

Rubrics

- 2 pnts for each correct answer.
- -1 pnt for wrong answer with reasonable explanation.
- 0 pnt for wrong answer without reasonable explanation.

1. The intersection of two convex sets is convex.
   True False

   True. Intersection preserves the convexity.

2. Given two convex sets $A_1, A_2 \subset R^n$, the set $A_3 = \{[x_1^T, x_2^T]^T \,|\, x_1 \in A_1,\ x_2 \in A_2\}$ in R^{2n} is also convex.
   True False

   True. We can prove the convexity by definition.

3. Given two convex sets $A_1, A_2 \subset R^n$, the set $A_3 = \{x_1 + x_2 |\ x_1 \in A_1,\ x_2 \in A_2\}$ is also convex.
   True False

   True. We can prove the convexity by definition.

4. Given two convex sets $A_1, A_2 \subset R$, the set $A_3 = \{x_1 \times x_2 |\ x_1 \in A_1,\ x_2 \in A_2\}$ is also convex.
   True False

   True. The sets are intervals in R.

5. Function $f(x) = -logx,\ x \in R_{++}$ is a convex function
   True False

   True. According to second-order condition the function is convex.

6. Function $g(y) = f(x|x = Ay)$ where matrix $A \in R^{mn}$ is used for an affine transformation from $y \in R^n$ to $x \in R^m$, is convex if $f(x)$ is convex.
   True False

   True. The convexity is preserved with affine operations.

7. Function $g(y) = \min_x f(x, y)$ is convex if $f(x, y)$ is differentiable.
   True False

   False. See Chap 3.2.5.

8. Function $g(y) = \min_x f(x,y)$ is convex if function $h(x) = \max_y f(x,y)$ is a convex function of input x.
True False

False. See Chap 3.2.5.

9. Minimization of function $f(x) = x_1^3 x_2 - x_3^2 x_4^5$ for $x \in R_+^4$ is a geometric programming problem.
True False

False. The coefficient should be positive in GP.

10. Given a convex function $f(x)$ for $x \in R^n$, the condition $\nabla f(\bar{x}) = 0$ implies that $\bar{x}$ is a solution either maximizing or minimizing the function.
True False

False. For convex function only minimizing.

2 Theorems and Proofs

Problem 2.1 State and prove the convexity of pointwise maximization of a set of convex functions. (10 points)
Rubrics

- -2 pnts for each incorrect/missing statement.
- -1 pnt for minor mistake.

Either using the definition or properties of epigraph to prove the convexity.

Problem 2.2 Show that the dual function yields lower bounds on the optimal value p^* of the primal problem, i.e. for any Lagrange multipliers $\lambda \geq 0$ and any ν, we have the dual function, $g(\lambda, \nu) \leq p^*$. (10 points)
Rubrics

- Points deducted if the statement is incorrect or incomplete.

Refer to Chap 5.1.3 or use saddle-point property.

3 Case Studies

Problem 3.1 Dual Cone: Given a cone $K = \{\theta_1 u_1 + \theta_2 u_2 \mid u_1 = [2,-1]^T,\ u_2 = [1,0]^T,\ \theta_1 \geq 0, \theta_2 \geq 0\}$, find the dual cone of K. (15 points)
Rubrics

- -5 pnts for partially correct answer with proper process.

Given a cone $K = \{A^T x | x \geq 0\}$, its dual cone $K^* = \{x | Ax \geq 0\}$.
The implicit format for the dual cone is $K^* = \{x| \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} x \geq 0\}$.
The explicit format $K^* = \{x_1 u_1 + x_2 u_2 \mid u_1 = [1,2]^T,\ u_2 = [0,-1]^T, x \geq 0\}$.

Problem 3.2 Conjugate Function: Given a function $f(x) = 2x_1^2 + 3(x_2 - 4)^2,\ x \in R^2$, find the conjugate function $f^*(y), y \in R^2$. (15 points)
Rubrics

- -5 pnts for partially correct answer with proper process.
- -2 pnts for minor mistake.

The conjugate function $f^*(y) = \frac{1}{8} y_1^2 + \frac{1}{12} y_2^2 + 4y_2$ for $y \in \mathbb{R}^2$.

Problem 3.3 Primal Dual Formulation: Given a linear programming problem,
minimize $f_0(x) = c^T x$
subject to $Ax \leq b$, and $Px = q$, where $x \in R^n$.
Derive the dual problem formulation. (10 points)
Rubrics

- -5 pnts for partially correct answer with proper process.
- -2 pnts for missing constraint in the dual problem.
- -1 pnt for minor mistake.

The Lagrangian with $\lambda, \nu \in R^n$ and $\lambda \geq 0$

$$\begin{aligned} L(x, \lambda, \nu) &= c^T x + \lambda^T (Ax - b) + \nu^T (Px - q) \\ &= -b^T \lambda - q^T \nu + (c + A^T \lambda + P^T \nu)^T x \end{aligned}$$

The dual function is

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = -b^T \lambda - q^T \nu + \inf_x \, (c + A^T \lambda + P^T \nu)^T x$$

which is bounded below only when $c + A^T \lambda + P^T \nu = 0$. We have

$$g(\lambda, \nu) = \begin{cases} -b^T \lambda - q^T \nu & \text{if } c + A^T \lambda + P^T \nu = 0, \ \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem is formulated as

$$\begin{aligned} \text{maximize} \quad & -b^T \lambda - q^T \nu \\ \text{subject to} \quad & c + A^T \lambda + P^T \nu = 0 \\ & \lambda \geq 0 \end{aligned}$$

4 Problems from Exercises

Problem 4.1 Prove the inequality $D(p,q) = \sum_{i=1}^{n} p_i log(p_i/q_i) - p_i + q_i \geq 0$ for all $p,q \in R^n_{++}$. (10 points)
Rubrics

- -2 pnts for each incomplete/incorrect statement.

See exercise 3.13.
Some common mistakes: if a function $f(x,y)$ is convex of x and convex of y individually, we could not derive that $f(x,y)$ is convex of (x,y). Consider the second-order condition, the Hessian is not guaranteed to be positive semi-definite if the diagonal terms are PSD. Off diagonal terms could cause negative eigenvalues of the matrix. Proof is required to show whether $\nabla^2 f(x,y)$ is PSD or not.

Problem 4.2 Consider a convex problem with no equality constraints,
minimize $f_0(x)$
subject to $f_i(x) \leq 0, i = 1,...,m$.
Assume that vector $x^* \in R^n$ and Lagrange multiplier λ^* satisfy the KKT conditions. Use KKT conditions to prove the following.
$\nabla f_0(x^*)^T(x - x^*) \geq 0$ for all feasible x. (10 points)
Rubrics

- -2 pnts for each incomplete/incorrect statement.
- -3 pnts for using ”KKT conditions $\Leftrightarrow$ primal and dual optimal solution” directly.

See exercise 5.31.
The objective is to show that KKT conditions could be interpreted as $\nabla f_0(x^*)^T(x - x^*) \geq 0$, which is the optimal criterion for convex problem, instead of deriving from the conclusion that KKT conditions are sufficient for the primal and dual optimal.