CSE 203B W22 Midterm 10AM 2/19/2022-10AM 2/22/2022

Submit your solution to gradescope before the due time.

Policy of the Exam: Here is the policy of the exam:

1. This is an open-book take-home exam. Internet search is permitted. However, you are required to work by yourself. Consultation or discussion with any other parties is not allowed.
2. You are not required to typeset your solutions. We do expect your writing to be legible and your final answers clearly indicated. Also, please allow sufficient time to upload your solutions.
3. You are allowed to check your answers with programs in Matlab, CVX, Mathematica, Maple, NumPy, etc. Be aware that these programs may not produce the intermediate steps needed to receive credit.
4. If something is unclear, state the assumptions that seem most natural to you and proceed under those assumptions. Out of fairness, we will not be answering questions about the technical content of the exam on Piazza or by email. The solution will then be graded based on the reasonable assumptions made.

Part I: True or False: Explain your answer with one sentence (27 pts)

I.1 (convex set): Set $\{(x^2, y^2)|x + y \geq 4, x, y \in R_+\}$ is convex.
T/F:

I.2 (dual cone): Given cone $K = \{x|Ax \geq 0, x \in R^n\}$, where $A \in R^{m\times n}$ its dual cone is $K^* = \{y|A^T y \geq 0, y \in R^m\}$.
T/F:

I.3 (Convex Function): Given a function $f(x) = log(e^{a_1x_1} + e^{a_2x_2} + e^{a_3x_3})$ with domain $D = \{x|x \in R^3\}$, we can show that $f(x)$ is a convex function for every arbitrary vector $a \in R^3$.
T/F:

I.4 (Conjugate Function): Given function $f(x) = x_1^2 - 4x_1x_2 + x_2^2$, where $x \in R^2$, then the conjugate of the conjugate function, $f^{**}(x)$, is equal to itself, i.e., $f^{**}(x) = f(x)$.
T/F:

I.5 (Convex Function): Function $g(x) = min_y f(x, y)$ is convex function, if $f(x, y)$ is a convex function with respect to variable x.
T/F:

I.6 (Convex Function): Given a differentiable but nonconvex function $f(x)$, where $x \in R^n$, and a fixed point $\bar{x} \in R^n$, the hyperplane

$$[\nabla f(\bar{x})^T, -1](\begin{bmatrix} x \\ t \end{bmatrix} - \begin{bmatrix} \bar{x} \\ f(\bar{x}) \end{bmatrix}) = 0$$

is a supporting hyperplane of epigraph, **epi** $f = \{ \begin{bmatrix} x \\ t \end{bmatrix} | f(x) \leq t \}$.
T/F:

I.7 (Problem Formulation): For every convex optimization problem defined as eq. (4.1) in textbook, where all functions are convex, there is always an optimal solution.
T/F:

I.8 (Problem Formulation/Duality): Given a convex programming problem:
minimize $f_0(x)$, subject to $Ax \leq b$, $x \in R^n$, $A \in R^{m \times n}$, $b \in R^m$,
where $f_0(x)$ is a differentiable convex function, we can claim that
$\nabla f_0(\bar{x}) \in \{-A^T\theta | \theta \in R^m_+\}$
is a necessary condition for $\bar{x}$ to be an optimal solution.
T/F:

I.9 (Duality): Given a function $f(x, y)$, the inequality
$\min_x \max_y - f(x, y) \geq \max_y \min_x - f(x, y)$
is always true.
T/F:

Part II: Problem Solving: Show your process

Problem 1. Dual Cone: Find the dual cone of the following cones. (20 pts)
1.1. $K = \{ \begin{bmatrix} x \\ t \end{bmatrix} | ||x||_p \leq t \}$, where $p \geq 1$.
1.2. $K = \{ \begin{bmatrix} x \\ t \end{bmatrix} | ||x||_p \leq t \}$, where $0 < p < 1$.

Problem 2. Conjugate Function: Find the conjugate function of the following functions. (20 pts)
2.1. $f(x) = x_1 x_2$, where $x \in R^2_+$.
2.2. $f(x) = \begin{cases} ||x||_p^p, & ||x||_p \leq a, \\ a^{1-\frac{1}{p}} ||x||_p, & ||x||_p > a, \end{cases}$
where variable $x \in R^n$, constants $a \in R_{++}$, and $p \geq 1$.

Problem 3. Graph Embedding. (33 pts)
Following Homework 4 Assignment II.2 Graph Embedding problem context and notations, our problem statement is the following.

$$\begin{array}{ll} \min_{x_1, y_1 \in \mathbb{R}^{n-k}} & x_1^\top L' x_1 + y_1^\top L' y_1 + b^\top x_1 + d^\top y_1. \\ \text{subject to} & \mathbf{1}^T x_1 = 0, \mathbf{1}^T y_1 = 0, \end{array}$$

where $\mathbf{1}$ is a vector of 1s, i.e. the sum of elements in x_1 is zero, and the sum of elements in y_1 is zero. Note that in contrast to Homework 4, we don't have quadratic constraints. Instead, we have linear equations.

3.1. Find the dual of the above problem

3.2. Solve this problem using the same data from homework 4.

"a partial framework in Python to get you started:
`https://colab.research.google.com/drive/1apgxNJGN1E4_W6awYbbhNxTyLOVvvMVH?usp=sharing`. If you prefer a different language, you can also download a .txt file containing L, x, y, and the indices of the fixed nodes:
`https://piazza.com/class_profile/get_resource/kx85xrdgigl5m5/kzfw6ud6fd964c`
(idx, x, y are the first 3 columns)"

3.3. Prove that your numerical solution is optimal.