Midterm Review for CSE 203B

Chester Holtz

Based on slides by Fangchen Liu
& Prof. Stephen Boyd

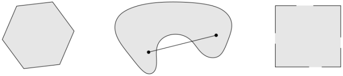
Logistics

- Released on course website: `http://cseweb.ucsd.edu/classes/wi21/cse203B-a/`
- Full 48 hours, submission on gradescope
- Released Tuesday 2/16 10:00 am PST, due Thursday 2/18 10:00 am PST
- 2 sections:
  - $\sim$ 10 True/False (with explanation)
  - $\sim$ 5 Derivations/simple proofs
  - At least one programming question
  - $\sim$ 70% based on homework questions
- No questions will be answered on piazza (sorry!)

Overview

- Convex sets
- Convex separation
- Convex functions
- Conjugate function
- Lagrangian Dual
- Logistics and other recommended topics

Convex sets: definition



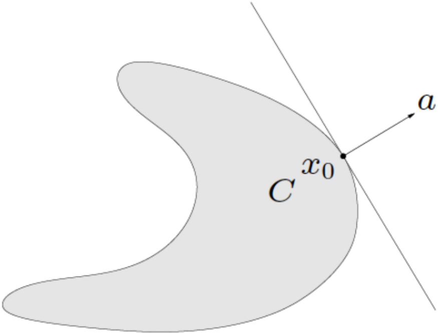
- A set $S \subseteq \mathbb{R}^d$ is convex if the line segment between any two points in C lies in C: for any $x_1, x_2 \in C$ and $0 \leq \theta \leq 1$, $\theta x_1 (1 - \theta) x_2 \in C$
- Example: the polytope $\mathcal{K} = \{x | Ax \leq b\}$ for $x, b \in \mathbb{R}^d$, $A \in \mathbb{R}^{m \times n}$

Supporting Hyperplane Theorem

A supporting hyperplane to a set C is defined with respect to a boundary point x_0:

$$\{x | a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$.

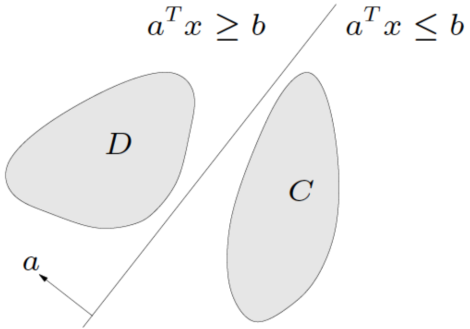


Supporting hyperplane theorem: If C is convex, then there exists a supporting hyperplane at every boundary point of C.

Separating Hyperplane Theorem

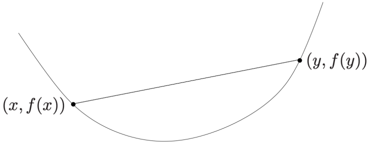
If C and D are nonempty disjoint convex sets, there exists $a \neq 0$ s.t.

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$



The hyperplane $\{x | a^T x = b\}$ separates C and D.
Strict separation requires additional assumptions (e.g. C is closed, D is a singleton).

Convex functions: definition

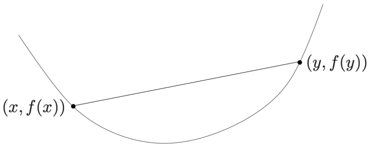


- A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\mathrm{dom} f$ is a convex set and if for all $x, y \in \mathrm{dom} f$ and $0 \leq \theta \leq 1$

  $$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y) \quad \text{Jensen's inequality}$$

- Concave functions: $-f$ is convex

Convex functions: definition

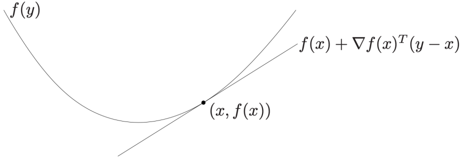


- A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom} f$ is a convex set and if for all $x, y \in \mathrm{dom} f$ and $0 \leq \theta \leq 1$

  $$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y) \quad \text{Jensen's inequality}$$

- Concave functions: $-f$ is convex

Convex functions: first order condition



- If f is differentiable ($\mathrm{dom} f$ is open, ∇f exists $\forall x \in \mathrm{dom} f$) then f is convex iff $\mathrm{dom} f$ is convex and for all $x, y \in \mathrm{dom}\ \mathrm{f}$

$$f(y) \geq f(x) + \nabla f(x)^T(y - x)$$

Convex functions: second order condition

- Suppose f is twice-differentiable ($\mathrm{dom} f$ is open and its Hessian exists $\forall x \in \mathrm{dom} f$) then f is convex iff $\mathrm{dom} f$ $\mathrm{dom} f$ is convex and for all $x, y \in \mathrm{dom\ f}$

$$\nabla^2 f \succcurlyeq 0 \quad \text{(positive semidefinite)}$$

Convex functions: establishing convexity

By definition

- Show by definition or first-order condition
- For twice-differentiable functions, show $\nabla^2 f \succeq 0$

By convexity-preserving operations

- Nonnegative weighted sum
- Composition with affine function
- Pointwise maximum and supremum
- Composition
- Minimization
- Perspective

Convex functions: establishing convexity

By definition

- Show by definition or first-order condition
- For twice-differentiable functions, show $\nabla^2 f \succcurlyeq 0$

By convexity-preserving operations

- Nonnegative weighted sum
- Composition with affine function
- Pointwise maximum and supremum
- Composition
- Minimization
- Perspective

Convex functions: relationship with convex sets

- A function is convex iff its epigraph is a convex set
- Consider a convex function f and $x, y \in \mathrm{dom} f$

$$t \geq f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

- The hyperplane supports $\mathrm{epi} f$ at $x, f(x)$, for any

$$\begin{aligned} & y, t \in \mathrm{epi} f \\ & \implies \nabla f(x)^T (y - x) + f(x) - t \leq 0 \end{aligned}$$

Convex functions: examples

powers of absolute value

$f = |x|^p$ is convex on $\mathbb{R}++$ and $p > 1$

Pf: Note that the composition of a convex and convex-increasing function is convex. Prove $|\cdot|$ is convex and x^p is convex and increasing.

TODO: Show log-convex function is convex $(g(x) = log(f(x))$, s.t. f convex. (first show for f twice-differentiable))

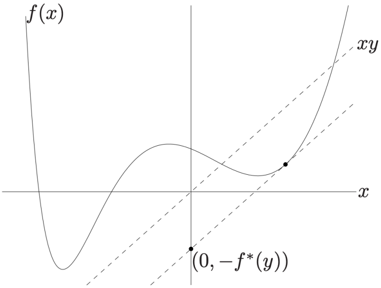
Convex functions: examples

quadratic form of inverse

$f : \mathbb{R}^n \times S^n \to \mathbb{R}$, $f(x, Y) = x^T Y^{-1} x$ is convex on $\mathbb{R}^n \times S^n_{++}$

Pf: Show epigraph of f is a convex set. Express epigraph as an LMI and apply the definiteness conditions of the Schur Complement (appendix 5.5).

Conjugate function



- Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the conjugate function

$$f^*(x) = \sup_{x \in \mathrm{dom} f} y^T x - f(x)$$

- $\mathrm{dom} f^*$ consists of $y \in \mathrm{dom} f$ such that $\sup_{y \in \mathrm{dom} f} y^T x - f(x)$ is bounded.
- $f^*(x)$ is convex even if $f(x)$ is not convex

Pointwise supremum

- If for each $y \in U$ $f(x, y)$ is convex in x, then

$$g(x) = \sup_{x \in U} f(x, y)$$

  is convex in x.
- Example: $f^*(x) = \sup_{x \in \mathrm{dom} f} y^T x - f(x)$
- Example: First order condition for convex functions

Duality

Primal problem

$$\min f_0(x)$$
$$f_i(x) \leq 0$$
$$h_i(x) = 0$$

Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \end{aligned}$$

Duality

Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$
\begin{aligned}
g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\
&= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)
\end{aligned}
$$

g is concave, can be unbounded for some $-\lambda$, ν.

Lower bound property

If $\lambda \geq 0$, then $g(\lambda, \nu) \leq p^*$.
proof: if $\bar{x}$ is feasible and $\lambda \geq 0$ then

$$
f_0(\bar{x}) \geq L(\bar{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(\bar{x}, \lambda, \nu) = g(\lambda, \nu)
$$

minimizing over all feasible $\bar{x}$ gives $p^* \geq g(\lambda, \nu)$.

Duality

Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \end{aligned}$$

g is concave, can be unbounded for some $-\lambda$, ν.

Lower bound property

If $\lambda \geq 0$, then $g(\lambda, \nu) \leq p^*$.
proof: if $\bar{x}$ is feasible and $\lambda \geq 0$ then

$$f_0(\bar{x}) \geq L(\bar{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(\bar{x}, \lambda, \nu) = g(\lambda, \nu)$$

minimizing over all feasible $\bar{x}$ gives $p^* \geq g(\lambda, \nu)$.

Duality example: Primal and Dual of an LP

$$\min_x c^T x$$
$$\text{s.t.} \quad Ax \leq 0$$

- The feasible set is the polytope $K = \{x | Ax \leq b\}$
- The Lagrange dual function of the primal problem is

$$g(\lambda) = \inf_x (c^T x + \lambda A x) = \begin{cases} 0 & A^T \lambda + c = 0, \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- The dual problem is

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & A^T \lambda + c = 0 \\ & \lambda \geq 0 \end{aligned}$$

Duality example: Primal and Dual of an LP

$$\min_x c^T x$$
$$\text{s.t.} \quad Ax \leq 0$$

- The feasible set is the polytope $K = \{x | Ax \leq b\}$
- The Lagrange dual function of the primal problem is

$$g(\lambda) = \inf_x (c^T x + \lambda A x) = \begin{cases} 0 & A^T \lambda + c = 0, \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- The dual problem is

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & A^T \lambda + c = 0 \\ & \lambda \geq 0 \end{aligned}$$

Duality example: Primal and Dual of an LP

$$\min_x c^T x$$
$$\text{s.t.} \quad Ax \leq 0$$

- The feasible set is the polytope $K = \{x | Ax \leq b\}$
- The Lagrange dual function of the primal problem is

$$g(\lambda) = \inf_x (c^T x + \lambda Ax) = \begin{cases} 0 & A^T\lambda + c = 0, \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- The dual problem is

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & A^T\lambda + c = 0 \\ & \lambda \geq 0 \end{aligned}$$

Duality example: Primal and Dual of an LP

$$\min_x c^T x$$
$$\text{s.t.} \quad Ax \leq 0$$

- The feasible set is the polytope $K = \{x | Ax \leq b\}$
- The Lagrange dual function of the primal problem is

$$g(\lambda) = \inf_x (c^T x + \lambda Ax) = \begin{cases} 0 & A^T\lambda + c = 0, \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- The dual problem is

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & A^T\lambda + c = 0 \\ & \lambda \geq 0 \end{aligned}$$

Duality example: Primal and Dual of an LP

$$\min_x c^T x$$
$$\text{s.t.} \quad Ax \leq 0$$

- The feasible set is the polytope $K = \{x | Ax \leq b\}$
- The Lagrange dual function of the primal problem is

$$g(\lambda) = \inf_x (c^T x + \lambda A x) = \begin{cases} 0 & A^T \lambda + c = 0, \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- The dual problem is

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & A^T \lambda + c = 0 \\ & \lambda \geq 0 \end{aligned}$$

- Farkas lemma: $Ax \leq 0, \quad c^T x < 0$ where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ is satisfied for some x iff $A\lambda = c$ s.t. $\lambda \geq 0$ has no solution.

Saddle point interpretation

- Max-min inequality for any f:
  $\sup_{z \in Z} \inf_{w \in W} f(w, z) \leq \inf_{w \in W} \sup_{z \in Z} f(w, z)$
- Now, consider the optimal values of the primal and dual problems:

$$p^* = \inf_x \sup_{\lambda \geq 0} L(x, \lambda) \geq \sup_{\lambda \geq 0} \inf_x L(x, \lambda)$$

Other

- Definitions and examples
- Duality
- Classification of convex problems: LP, GP, SOCP, QCQP, etc.
- CVXPY & Python