CSE 203B: Convex Optimization Week 3 Discuss Session

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- **Convex functions** (Ref. Chap.3)
  - Definition
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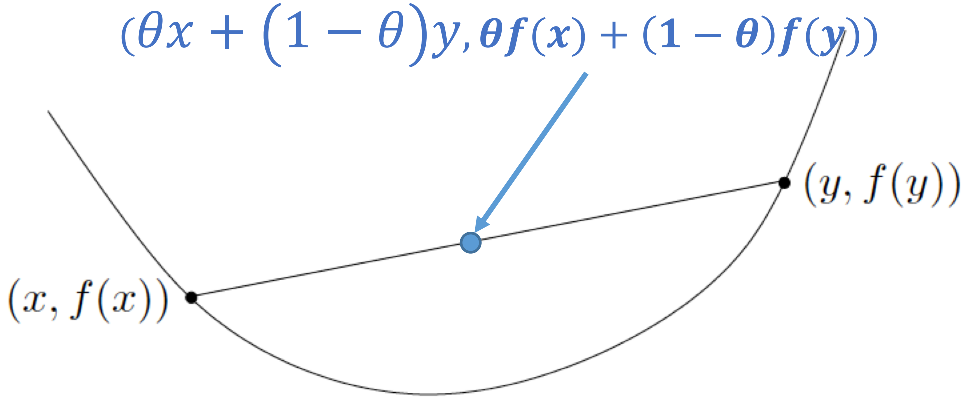
Definition of Convex Functions

- A function $f: R^n \to R$ is convex if $\mathrm{dom}\, f$ is a convex set and if for all $x, y \in \mathrm{dom}\, f$ and $0 \leq \theta \leq 1$
$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$
sometimes called Jensen's inequality
- Review the proof in class: necessary and sufficiency
- Strict convexity: $f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y), x \neq y, 0 < \theta < 1$
- Concave functions: $-f$ is convex



Restriction of a convex function to a line

- A function $f: R^n \rightarrow R$ is convex **if and only if** the function $g: R \rightarrow R$,

$$g(t) = f(x + tv), \qquad \mathrm{dom}\ g = \{t |\ x + tv \in \mathrm{dom}\ f\}$$

  is a convex on its domain for $\forall x \in \mathrm{dom}\ f, v \in R^n$.

- The property can be useful to check the convexity of a function

Example: Prove $f(X) = \log\det X$, $\mathrm{dom}\ f = S_{++}^n$ is concave.

Restriction of a convex function to a line

Example: Prove $f(X) = \log\det X$, $\operatorname{dom} f = S_{++}^n$ is concave.

Consider an arbitrary line $X = Z + tV$, where $Z \in S_{++}^n, V \in S^n$. Define $g(t) = f(Z + tV)$ and restrict g to the interval values of t for $Z + tV \succ 0$. We have

$$g(t) = \log\det(Z + tV) = \log\det(Z^{\frac{1}{2}}\left(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}}\right)Z^{1/2})$$

$$= \sum_{i=1}^{n} \log(1 + t\lambda_i) + \log\det Z$$

- Properties from HW0
- $\det AB = \det A \det B$
- $\det A = \prod_{i=1}^{n} \lambda_i$

where λ_i are the eigenvalues of $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$. So we have

$$g'(t) = \sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i}, \qquad g''(t) = -\sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + t\lambda_i)^2} \leq 0$$

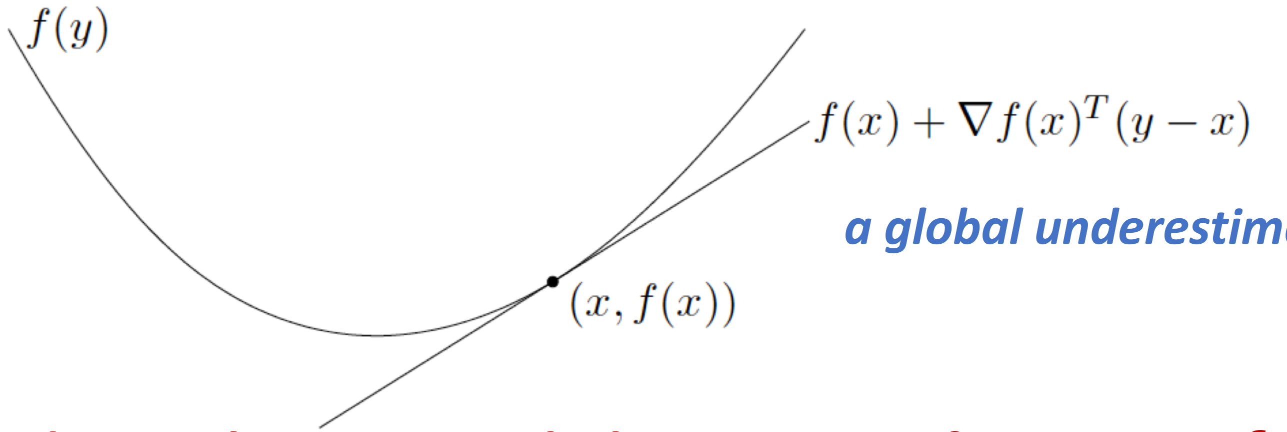
$g(t)$ is concave, hence $f(X)$ is concave. For more practice, see Exercise 3.18.

First-order Condition

- Suppose f is differentiable ($dom\ f$ is open and ∇f exists at $\forall x \in dom\ f$), then f is convex **iff** $\mathrm{dom}\ f$ is convex and for all $x, y \in dom\ f$

$$f(y) \geq f(x) + \nabla f(x)^T(y - x)$$

- Review the proof in class: necessary and sufficiency
- Strict convexity: $f(y) > f(x) + \nabla f(x)^T(y - x), x \neq y$
- Concave functions: $f(y) \leq f(x) + \nabla f(x)^T(y - x)$



a global underestimator

Proof of first-order condition: chap 3.1.3 with the property of restricting f to a line.

Second Order Condition

- Suppose f is twice differentiable ($dom\ f$ is open and its Hessian exists at $\forall x \in dom\ f$), then f is convex **iff** $\mathrm{dom}\ f$ is convex and for all $x, y \in dom\ f$

$$\nabla^2 f(x) \succcurlyeq 0 \text{ (positive semidefinite)}$$

- Review the proof in class: necessary and sufficiency
- Strict convexity:$\nabla^2 f(x) \succ 0$
- Concave functions: $\nabla^2 f(x) \preccurlyeq 0$

Example of Convex Functions

- Quadratic over linear function

$$f(x,y) = \frac{x^2}{y}, \text{for } y > 0$$

Its gradient $\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{2x}{y} \\ \frac{-x^2}{y^2} \end{bmatrix}$

Hessian $\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \frac{2}{y^3} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} \succcurlyeq 0 \Rightarrow$ convex

Positive semidefinite? $\begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^{\mathrm{T}}$, for any $u \in R^2, u^T(vv^T)u = (v^T u)^T (v^T u) = ||v^T u||_2^2 \geq 0.$

More examples see chap. 3.1.5

Epigraph

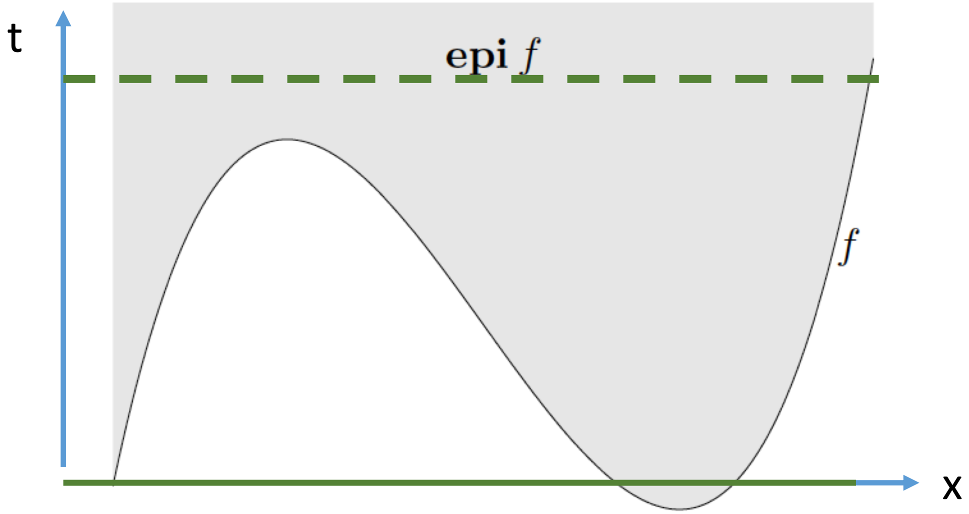
- α-*sublevel* set of $f: R^n \to R$

$$C_\alpha = \{x \in dom\ f |\ f(x) \leq \alpha\}$$

sublevel sets of a convex function are convex for any value of α.

- Epigraph of $f: R^n \to R$ is defined as

$$\mathbf{epi}\ f = \{(x, t) | x \in dom\ f, f(x) \leq t\} \subseteq \boldsymbol{R^{n+1}}$$



- A function is convex **iff** its epigraph is a convex set.

Relation between convex sets and convex functions

- A function is convex **iff** its epigraph is a convex set.
- Consider a convex function f and $x, y \in dom\ f$

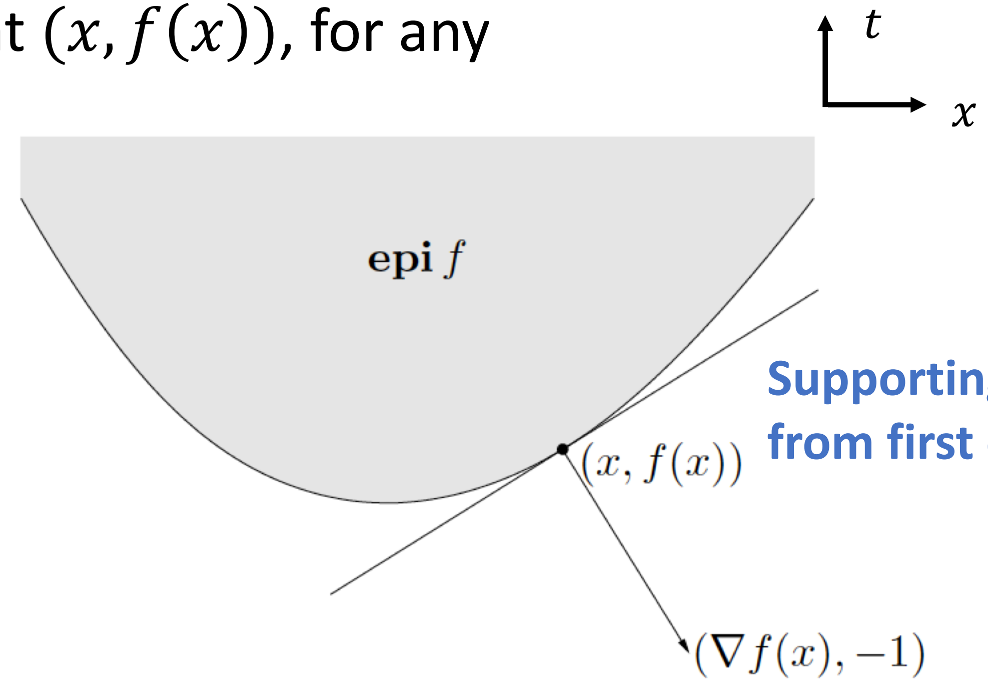
$$t \geq f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

epi *f* **First order condition for convexity**

- The hyperplane supports **epi *f*** at $(x, f(x))$, for any

$$(y, t) \in \mathbf{epi}\ f \Rightarrow$$
$$\nabla f(x)^T (y - x) + f(x) - t \leq 0$$
$$\Rightarrow \begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}^T \left(\begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0$$

normal vector of the supporting hyperplane



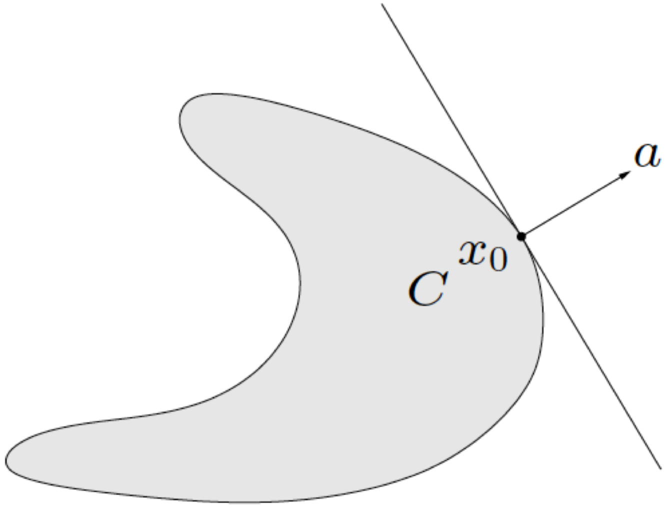
Supporting hyperplane, derived from first order condition

Recap: Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Reference

[1] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, http://stanford.edu/ boyd/cvxbook/, 2004.