W21 CSE 203B: Convex Set

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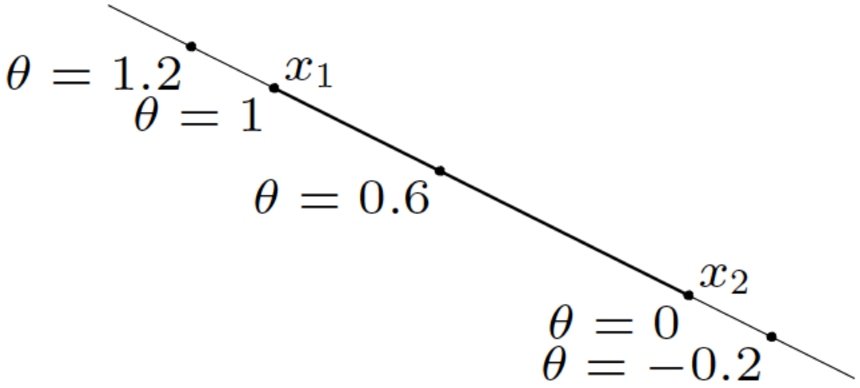
Affine set

line through x_1, x_2: all points

$$x = \theta x_1 + (1 - \theta) x_2 \qquad (\theta \in \mathbf{R})$$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)

Convex set

line segment between x_1 and x_2: all points

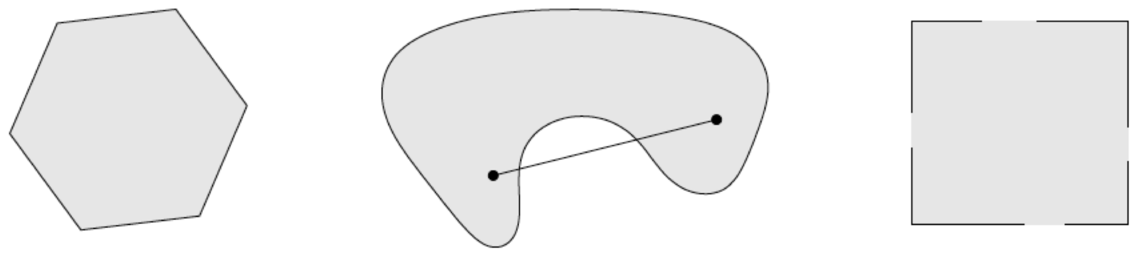
$$x = \theta x_1 + (1 - \theta) x_2$$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1 - \theta) x_2 \in C$$

examples (one convex, two nonconvex sets)



Convex combination and convex hull

convex combination of $x_1, \ldots, x_k$: any point x of the form

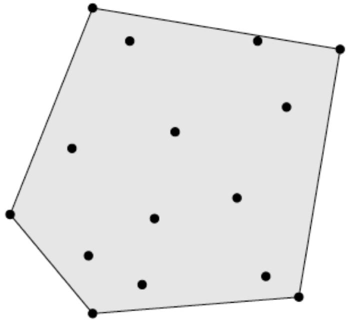
$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

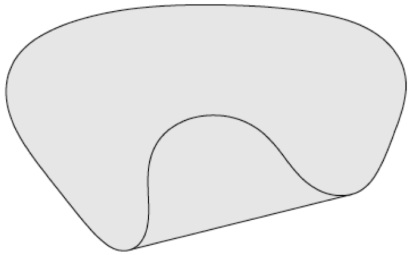
with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{conv}\, C$: set of all convex combinations of points in C

$$\mathbf{conv}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C,\ \theta_i \geq 0,\ i = 1, \ldots, k,\ \theta_1 + \cdots + \theta_k = 1\}.$$

Explicit expression of convex hull





The convex hull is the **smallest** convex set that contains C

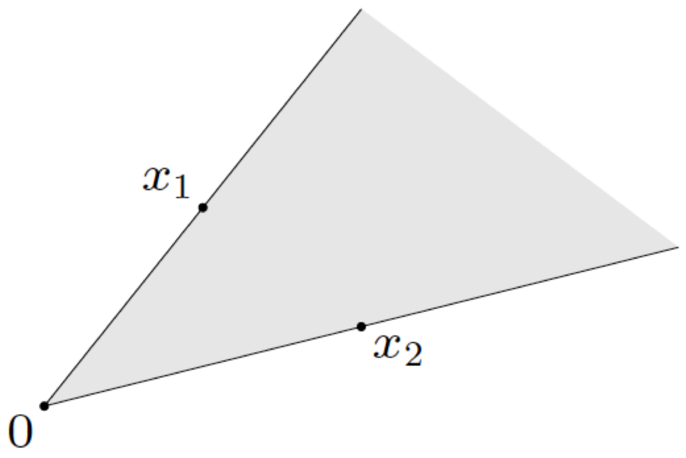
Cone

conic (nonnegative) combination of x_1 and x_2: any point of the form

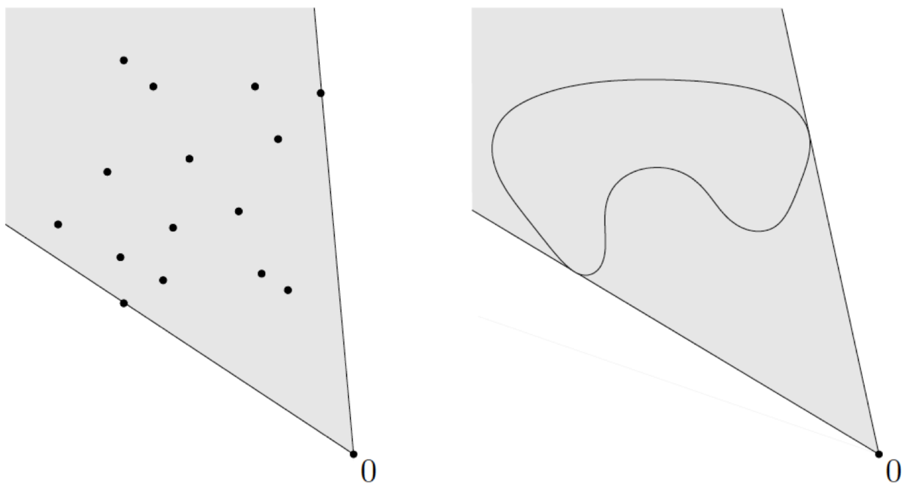
$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

A cone formed with x_1, x_2



Conic hull of sets



convex cone: set that contains all conic combinations of points in the set

Set Specification via Implicit or Explicit Enumeration

<u>Implicit Expression</u> $S_I = \{x|Ax \leq b, x \in R^n\}$

<u>Explicit Enumeration</u> $S_E = \{Ax \mid x \in R^n\}$

Implicit Expression:
Constraints
Min $f_o(x)$
Subject to
$Ax \leq b, x \in R^n$

Explicit Expression:
Enumeration
Min $f_o(Ax), x \in R^n$

Contents

- Example of convex sets
  - Simple examples
  - Hyperplanes and halfspaces
  - Prove the convexity of sets with operations that preserve the convexity

Examples of convex sets

- Empty set, point, line.
- Norm ball: $\{x : \|x\| \leq r\}$, for given norm $\|\cdot\|$, radius r.
- Hyperplane: $\{x : a^T x = b\}$, for given a, b.
- Halfspace: $\{x : a^T x \leq b\}$.
- Affine space: $\{x : Ax = b\}$, for given A, b.
- Polyhedron: $\{x : Ax \leq b\}$, where $\leq$ is interpreted componentwise. The set $\{x : Ax \leq b, Cx = d\}$ is also a polyhedron.
- Simplex: special case of polyhedra, given by $\text{conv}\{x_0, ..., x_k\}$, where these points are affinely independent. The canonical example is the probability simplex,

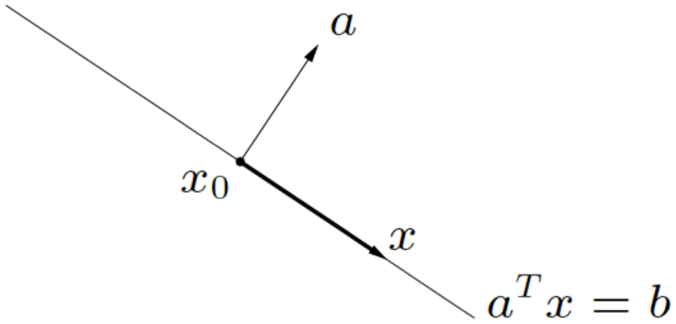
$$\text{conv}\{e_1, ..., e_n\} = \{w : w \geq 0, 1^T w = 1\}$$

Exercises:

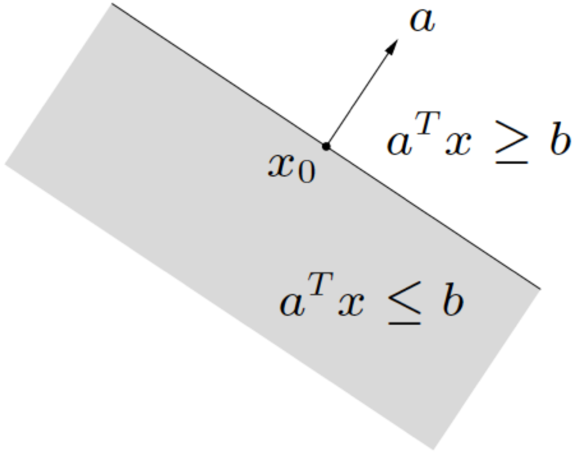
1) **Prove the convexity of the sets using the definition.**
2) **The above convex sets are described with implicit expression, find the explicit enumeration.**

Hyperplane and Halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

Hyperplane and Halfspaces

- Example of plane in $\mathbb{R}^3$: $3x + 4y + 6z = 12$
- Implicit express of the hyperplane

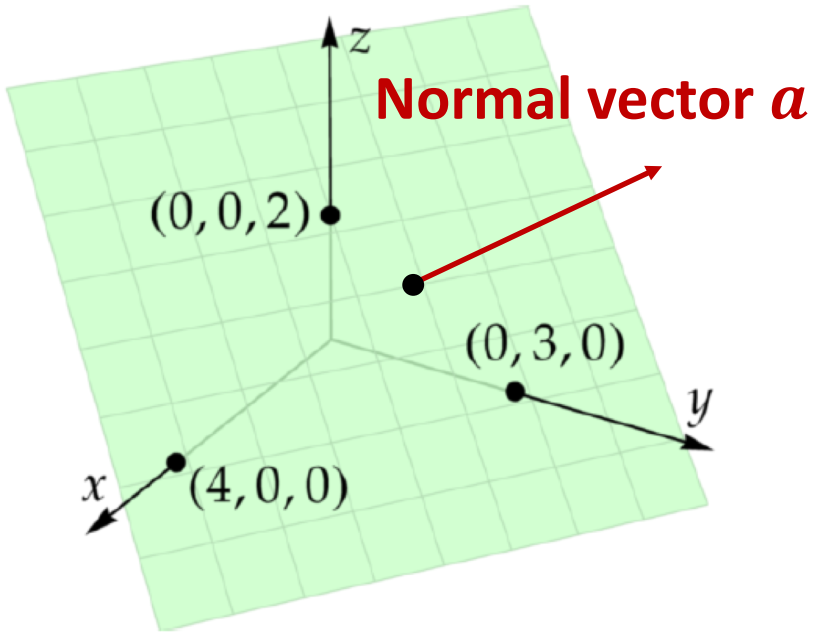
$$\{x \mid a^T x = b\} \text{ where } a^T = [3,4,6], b = 12$$

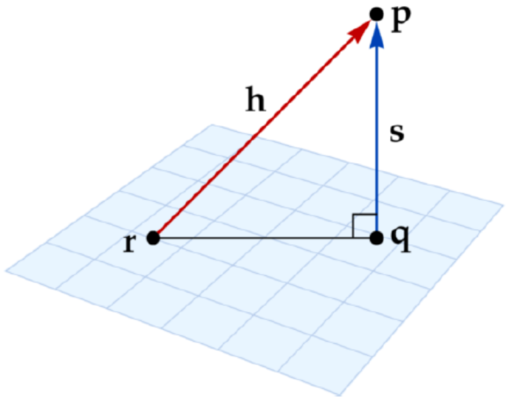
- Let x_0 be any point in the hyperplane, then the expression becomes

$$\{x \mid a^T(x - x_0) = 0\} = x_0 + a^{\perp}$$

Q: Write the explicit expression of the hyperplane.

- Find the distance from a point p to the plane.



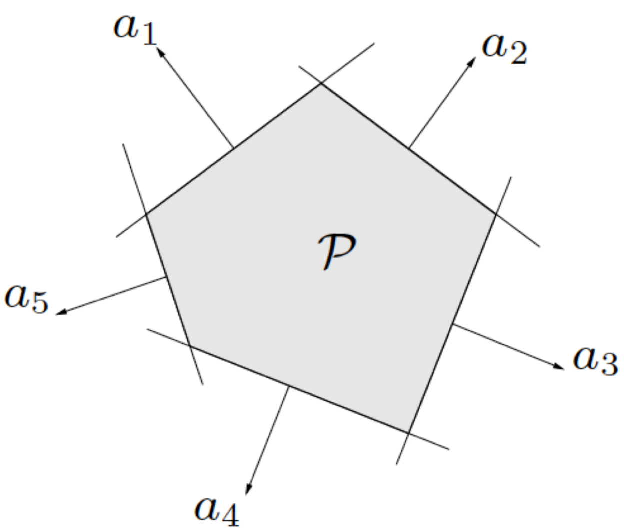


Prove the convexity of polyhedron

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

($A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is componentwise inequality)



Use the definition of convex set:

$$\boldsymbol{x_1, x_2 \in C, 0 \le \theta \le 1}$$
$$\boldsymbol{\Rightarrow \theta x_1 + (1-\theta) x_2 \in C}$$

Proof:

pick any two points in the polyhedron $\boldsymbol{x_1, x_2 \in P}$, for $0 \le \theta \le 1$ is the point

$$\boldsymbol{\theta x_1 + (1-\theta) x_2 \in P} \text{ ?}$$

polyhedron is intersection of finite number of halfspaces and hyperplanes

Positive semidefinite cone

notation:

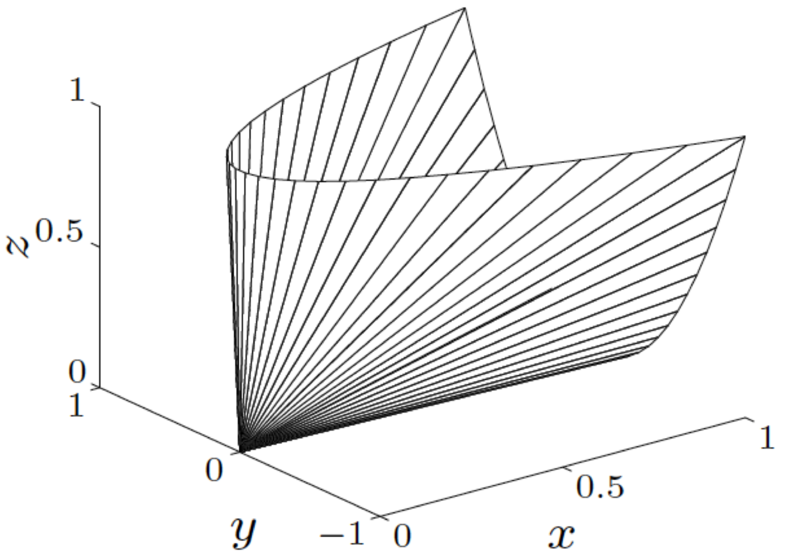
- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \iff z^T X z \geq 0 \text{ for all } z$$

  $\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$



Intersections preserve convexity

- Polyhedron: intersection of halfspaces and hyperplanes (convex)

$\Rightarrow$ *Is the intersection of a finite number of convex sets still convex?*

prove with the definition of convex sets.

Operations that preserve convexity:

- Intersection
- Affine function
- Perspective functions
- Linear-fractional functions

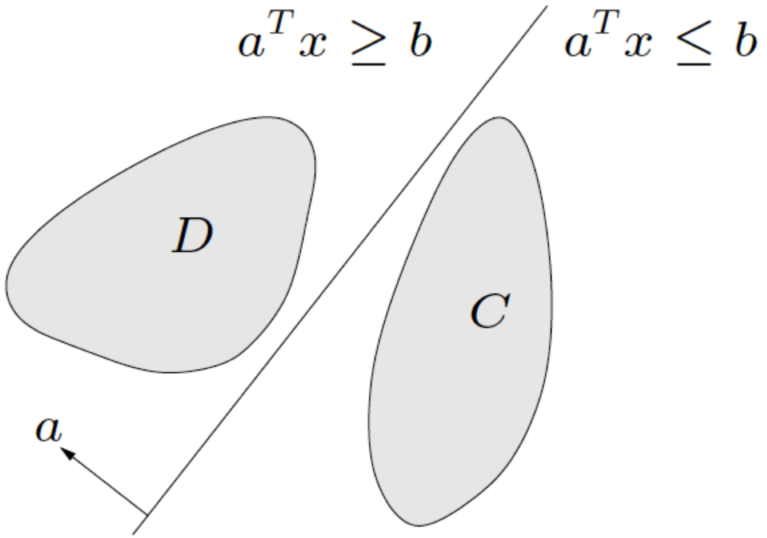
Contents

- Key properties of convex sets
  - Separating hyperplane theorem
  - Supporting hyperplane theorem

Separating hyperplane theorem

if C and D are nonempty disjoint convex sets, there exist $a \neq 0$, b s.t.

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

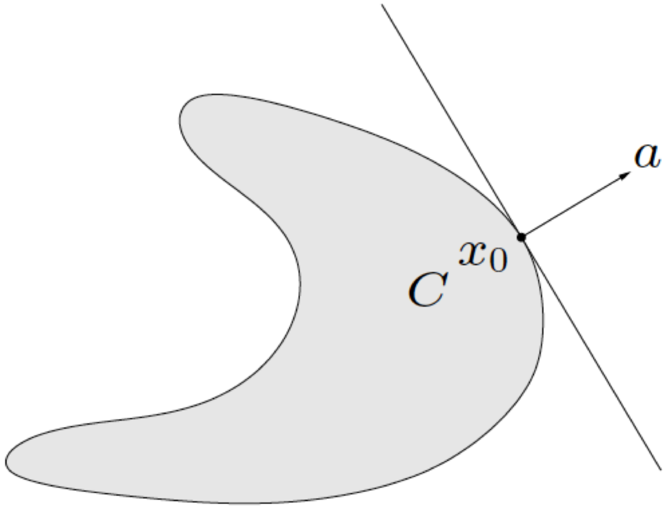
strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

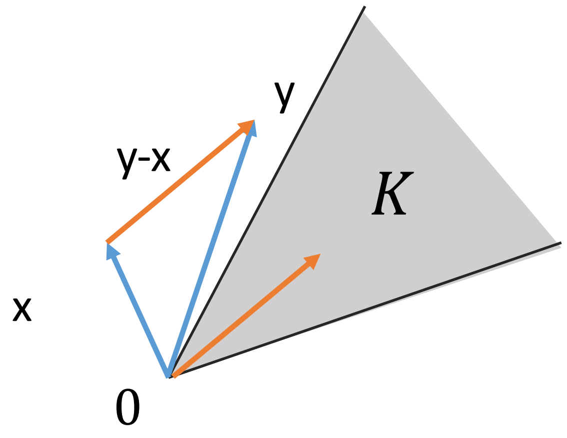
Contents

- Proper cone and generalized inequality
- Dual cone and generalized inequality

Generalized inequality

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)



generalized inequality defined by a proper cone K:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K, \qquad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \textbf{int}\, K$$

Review the properties of proper cones (chap 2.4.1) and generalized inequality (exercise 2.30).

Generalized inequality

examples

- componentwise inequality ($K = \mathbf{R}_+^n$) ***Nonnegative orthant***

$$x \preceq_{\mathbf{R}_+^n} y \iff x_i \le y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbf{S}_+^n$) ***Positive semidefinite cone***

$$X \preceq_{\mathbf{S}_+^n} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\le$ on $\mathbf{R}$, *e.g.*, ***a partial ordering on R^n analogy to ordinary inequality***

$$x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v$$

Minimum and minimal element

$\preceq_K$ is not in general a *linear ordering*: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

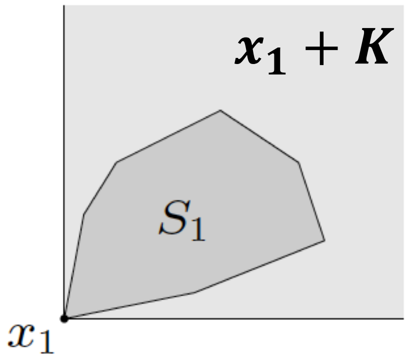
$$y \in S \quad \Longrightarrow \quad x \preceq_K y \qquad \Leftrightarrow \boldsymbol{S \subseteq x + K}$$

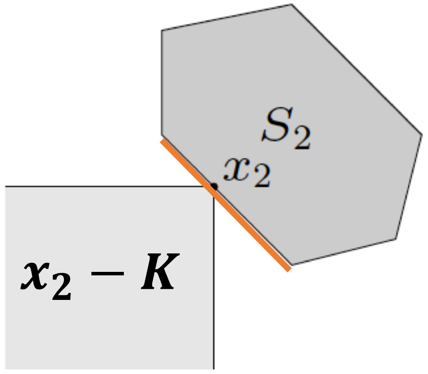
$x \in S$ is **a minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \quad \Longrightarrow \quad y = x \qquad \Leftrightarrow \boldsymbol{(x - K) \cap S = \{x\}}$$

example $(K = \mathbf{R}^2_+)$

x_1 is the minimum element of S_1
x_2 is a minimal element of S_2

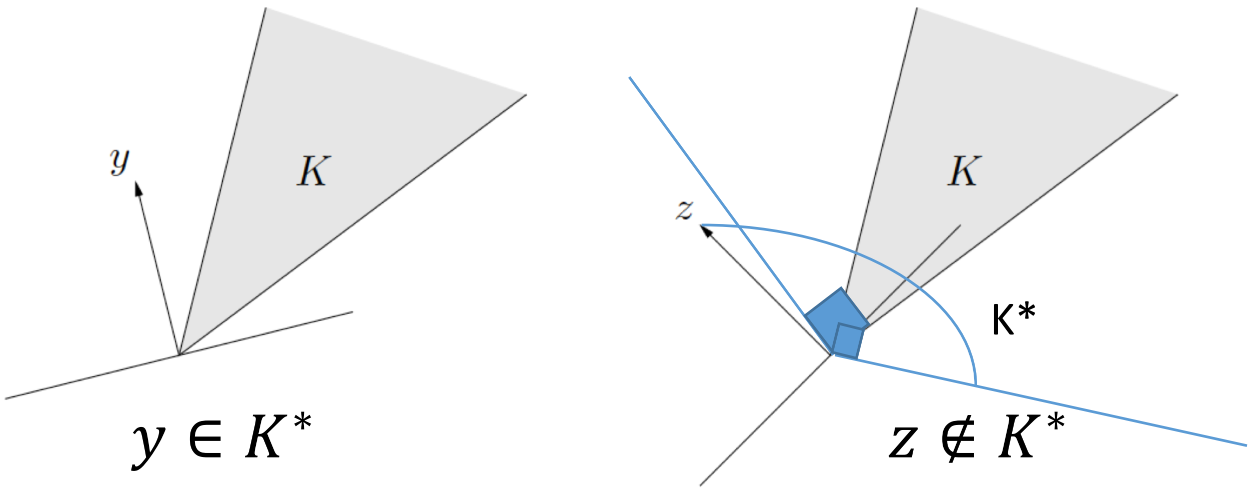




Dual cones

dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$



$y \in K^*$ $\qquad$ $z \notin K^*$

Find the **dual cones** of

- Subspace $V \in \boldsymbol{R}^n$
- Nonnegative orthant $K = \boldsymbol{R}_+^n$
- Positive semidefinite cone $K = \boldsymbol{S}_+^n$

Review the properties of dual cones (exercise 2.31).

reference

[1] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, http://stanford.edu/ boyd/cvxbook/, 2004.