CSE 203B: Convex Optimization

Discussion Week 3

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Content

- Qualification & Enumeration
- Dual Cone
- Support Vector Machine

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Qualification & Enumeration

Implicit/qualification expression:

$$\{\, x | Ax \leq b, x \in \ R^n\}$$

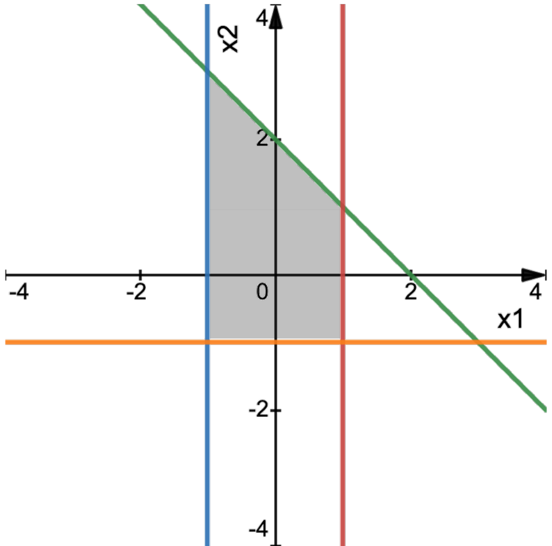
Explicit/enumeration expression:

$$\{\, U\theta | 1^T\theta = 1, \theta \in R_+^m\}$$

Example:

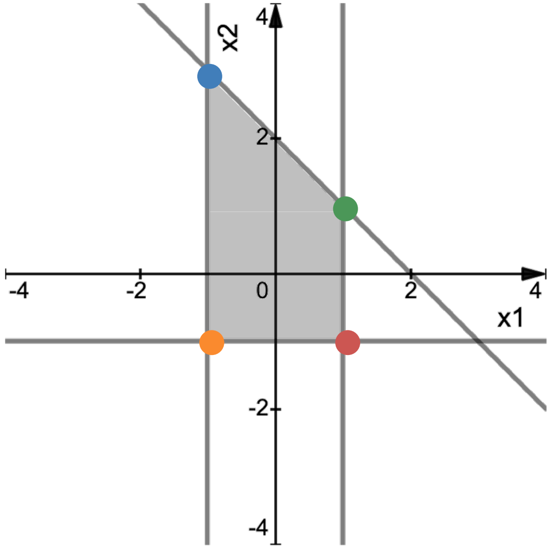
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$



Convert

(HW1.1: find extreme points)



Qualification & Enumeration

Implicit/qualification expression:

$$\{\, x | Ax \leq b, x \in \ R^n\}$$

Explicit/enumeration expression:

$$\{\, U\theta | 1^T\theta = 1, \theta \in R_+^m\}$$

Question:

The dimension m of $\theta \in \ R_+^m$ is determined by?

The number of extreme points in$\{\, x | Ax \leq b, x \in \ R^n\}$

Qualification & Enumeration

- HW 1.1 – how to find extreme points in a linear system of inequalities

  - **Extreme points**: Let $C \subseteq R^n$ be a convex set. For a $z \in C$, it is an extreme point of C if there is no distinct $x, y \in C$ and $\lambda \in (0,1)$ such that $z = (1-\lambda)x + \lambda y$.

    (always be the endpoint if in a line segment inside C)

Qualification & Enumeration

- HW 1.1 – how to find extreme points in a linear system of inequalities
  - Set n of the $m + n$ inequalities to equality (by turns) and solve the corresponding $n \times n$ system
  - Check whether the solution satisfies the remaining inequalities, discard if doesn't
  - Note $x \in R_+^6$, so $x_i \geq 0$ are also inequality conditions

https://web.math.utk.edu//~freire/linprog2005.pdf

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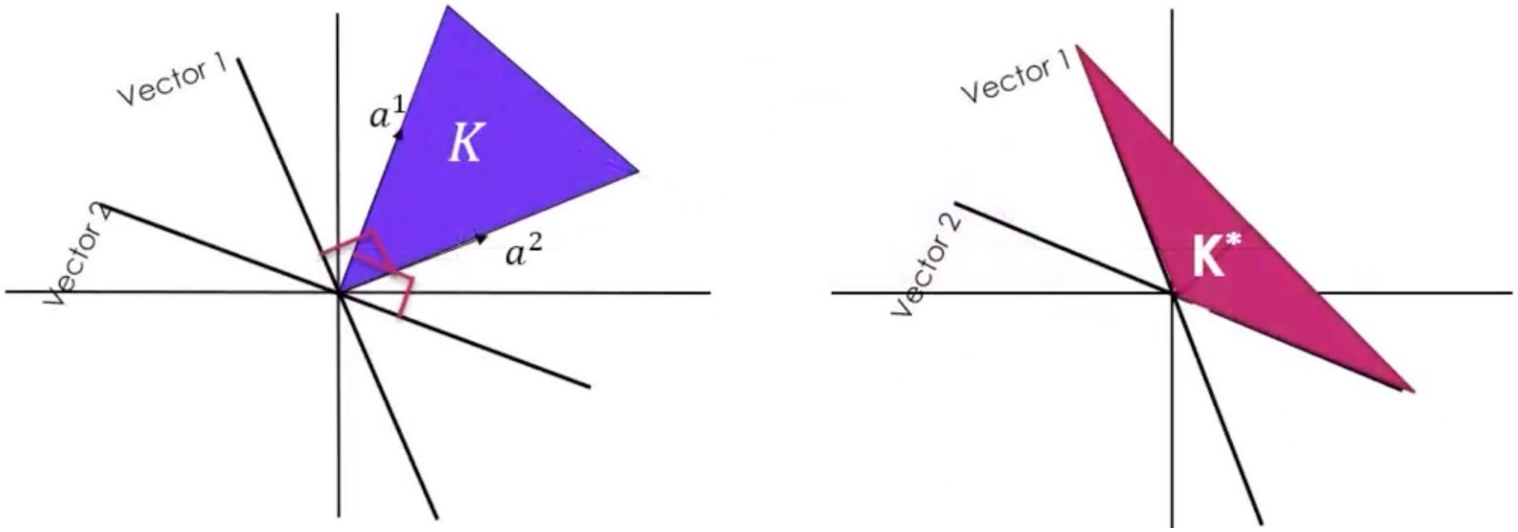
Dual Cone

- Let K be a cone, the set

$$K^* = \{ y \mid x^T y \geq 0, \forall x \in K \}$$

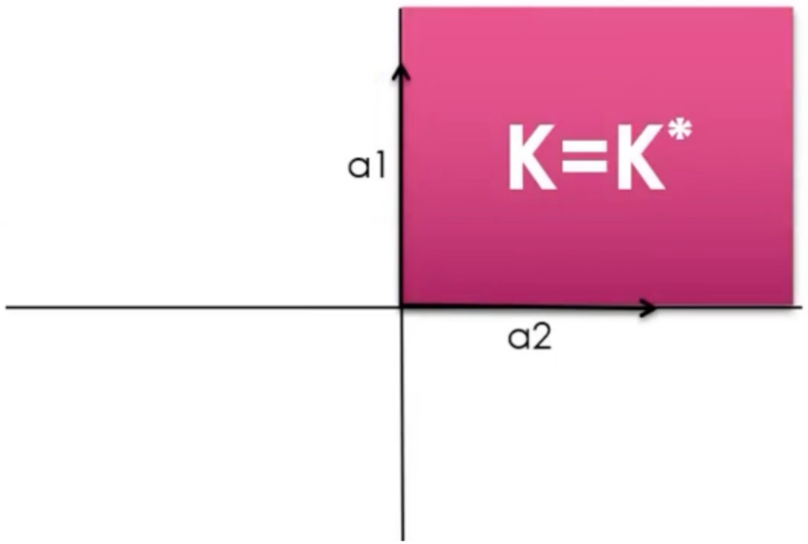
is called dual cone of K, K^* is **always convex**, even when the original K is not.



Dual Cone

- Example: **Self-dual**, the cone R_+^n is its own dual

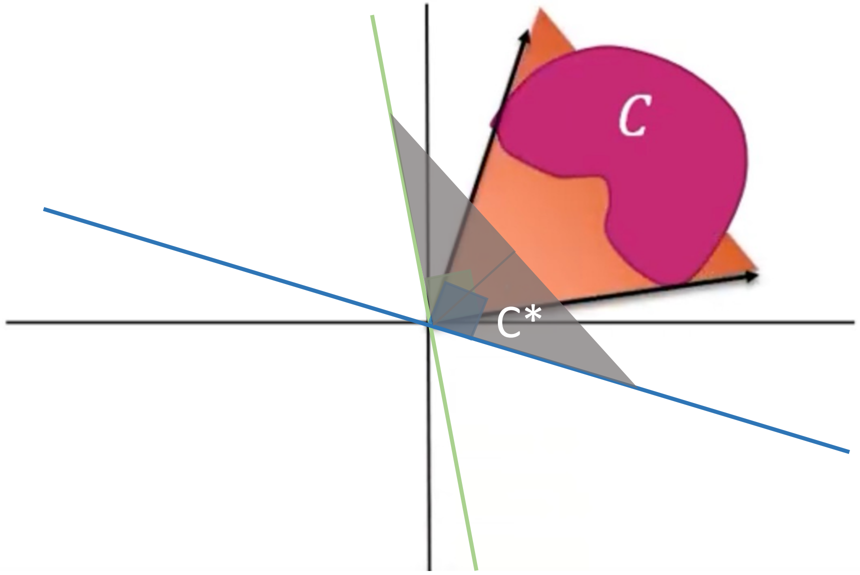
$$x^T y \geq 0, \forall x \geq 0 \Leftrightarrow y \geq 0$$



Dual Cone

- Example: Dual cone of a non-convex set is convex.

$$C^* = \{\, y \mid \langle y, x \rangle \geq 0, \forall x \in C \}$$



https://www.youtube.com/watch?v=CTNFqM8nRS0

Dual Cone

- HW 1.3

Derive the dual cone of the set $\{x|Ax \leq 0, x \in R^6\}$

for any y in the dual cone, there should be $y^T x \geq 0, \forall x \in \{x|Ax \leq 0, x \in R^6\}$

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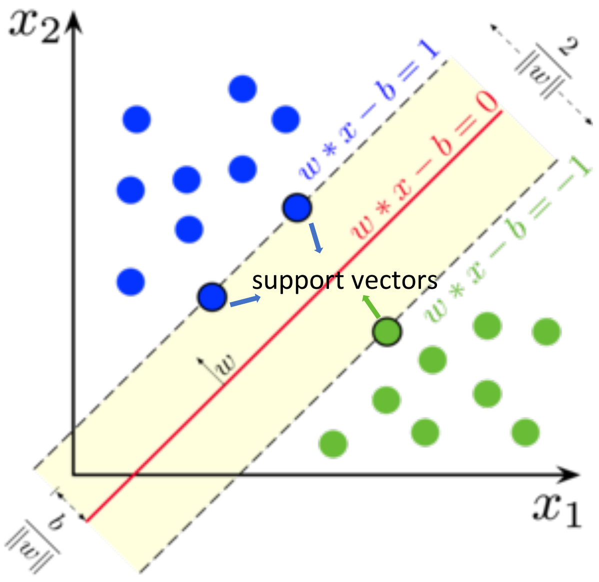
Support Vector Machine (SVM)

- Given a dataset of n points in form $(x_1, y_1), \ldots, (x_n, y_n)$ where class labels y_i are either 1 or -1. The SVM tries to find the "maximum-margin hyperplane" that divides the points into correct groups.

$$\min ||w||_2^2$$
$$s.t.\ \ w^T x_i - b \leq -1, if\ y_i = -1$$
$$w^T x_i - b \geq 1, if\ y_i = 1$$

$\Rightarrow$

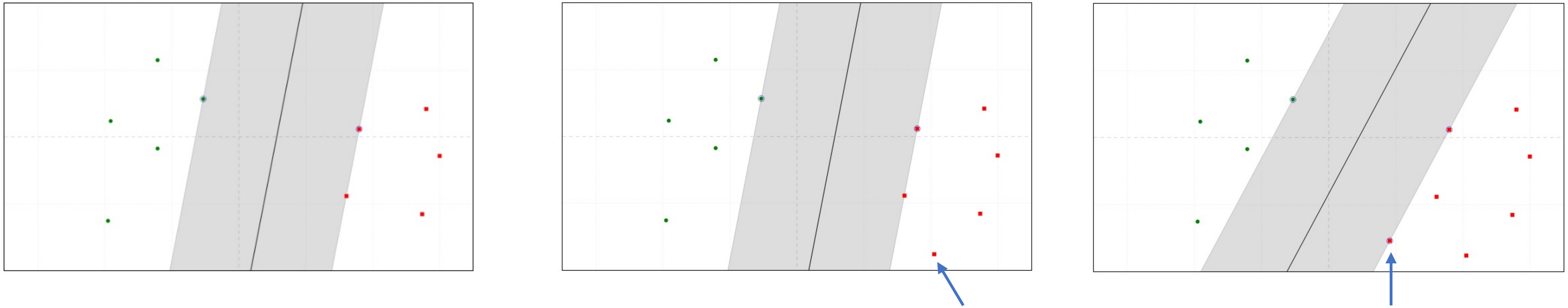
$$\min ||w||_2^2$$
$$s.t.\ \ (w^T x_i - b) y_i \geq 1$$



Geometrically, the distance between hyperplane $w^T x - b = 1$ and $w^T x - b = -1$ is $\frac{2}{||w||}$, so maximize it is equivalent to minimize $||w||$

Support Vector Machine (SVM)

- In hard-margin SVM, the decision boundary is only affected by the support vectors

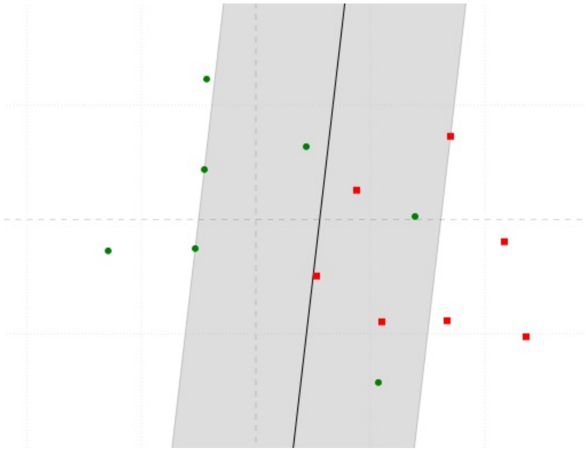


Check out this live demo

Support Vector Machine (SVM)

- Sometimes data are not linearly separable ☹
  - Soft-margin SVM: Add a loss function to penalize points on the wrong side

$$\max(0, 1 - y_i(w^T x_i - b))$$



if a point is on the wrong side, the loss is proportional to its distance to the margin. Minimize:

$$\lambda \|\mathbf{w}\|^2 + \left[\frac{1}{n}\sum_{i=1}^{n} \max\left(0, 1 - y_i(\mathbf{w}^T\mathbf{x}_i - b)\right)\right]$$

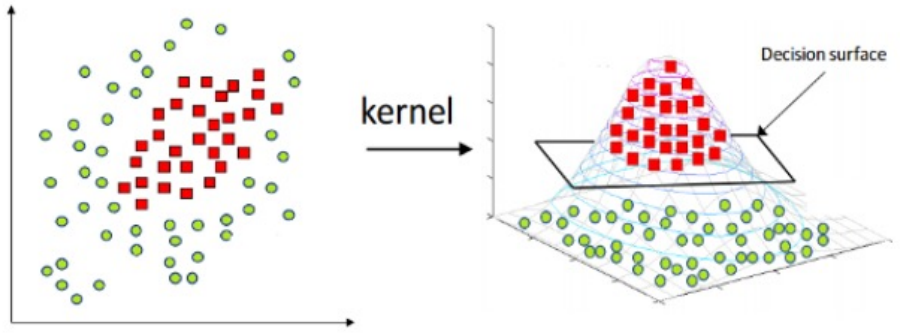
Support Vector Machine (SVM)

- Sometimes data are not linearly separable ☹
  - Kernel Trick
    - The idea of mapping the data to a higher dimensional space
    - Kernel functions is computational efficient
    - Common kernel functions:
      - Polynomial kernel

$$k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y} + 1)^d$$

      - Gaussian kernel

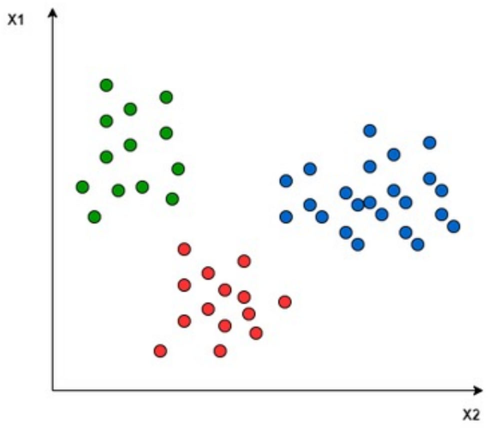
$$k(\mathbf{x}, \mathbf{y}) = e^{-\gamma \|\mathbf{x}-\mathbf{y}\|^2}, \gamma > 0$$

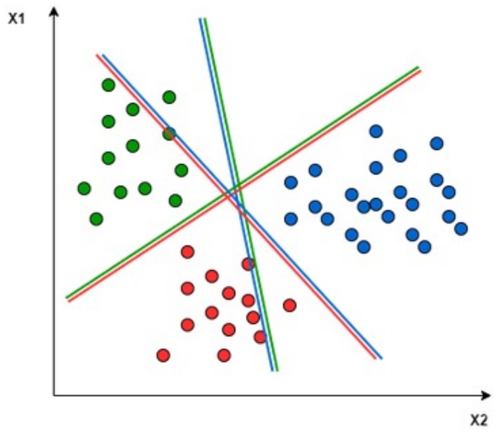


https://medium.com/@zxr.nju/what-is-the-kernel-trick-why-is-it-important-98a98db0961d

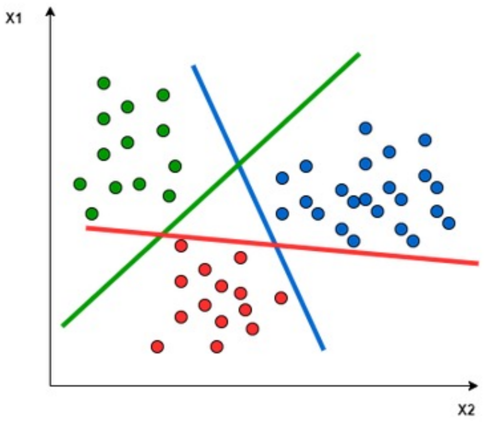
Support Vector Machine (SVM)

- Multiclass classification using SVM
  - One-to-one: use a binary SVM for each pair of classes
  - One-to-rest: separate a class and the rest of points





One-to-one



One-to-rest

https://www.baeldung.com/cs/svm-multiclass-classification

Thank you