CSE 203B

Discussion II

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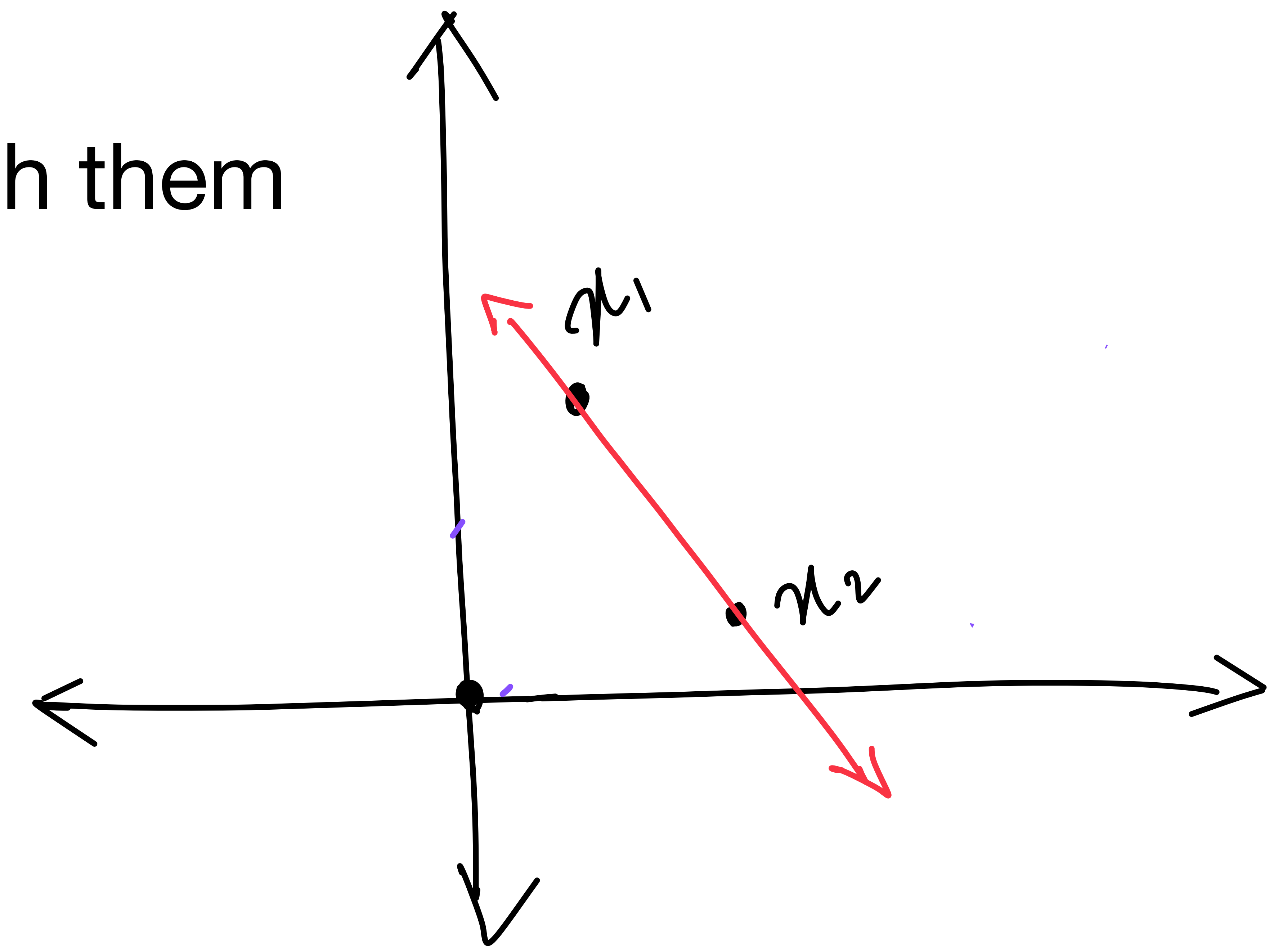
Contents

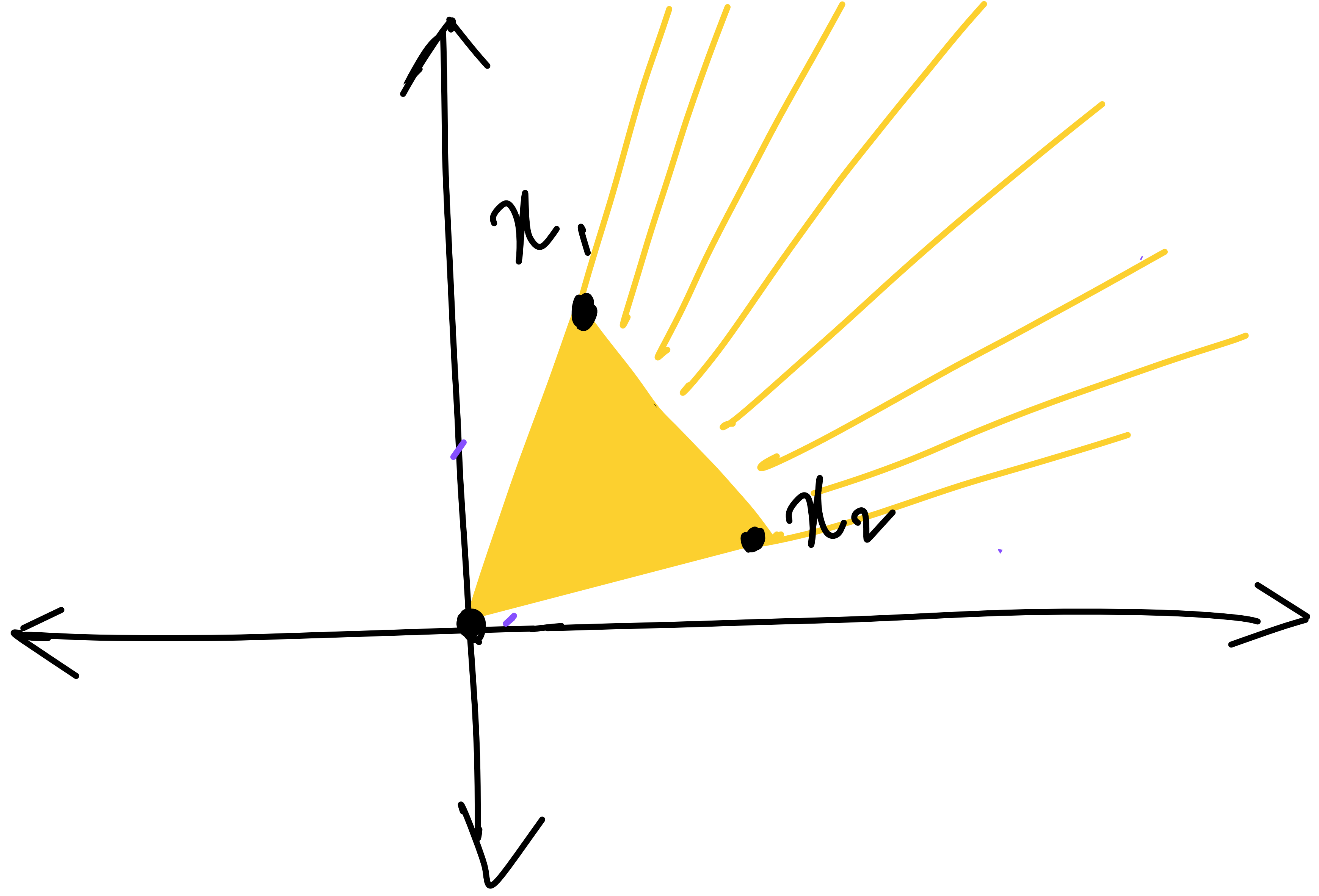
Affine set

- A collection of points $x_1, \cdots, x_n$ and $\theta_1, \theta_2, \cdots, \theta_n \in \mathbb{R} \mid \sum_{i=1}^{n} \theta_i = 1$ induces the affine set $\theta_1 x_1 + \cdots + \theta_n x_n$
- E.g. $x_1 \neq x_2 \in \mathbb{R}^2$ generates the line through them



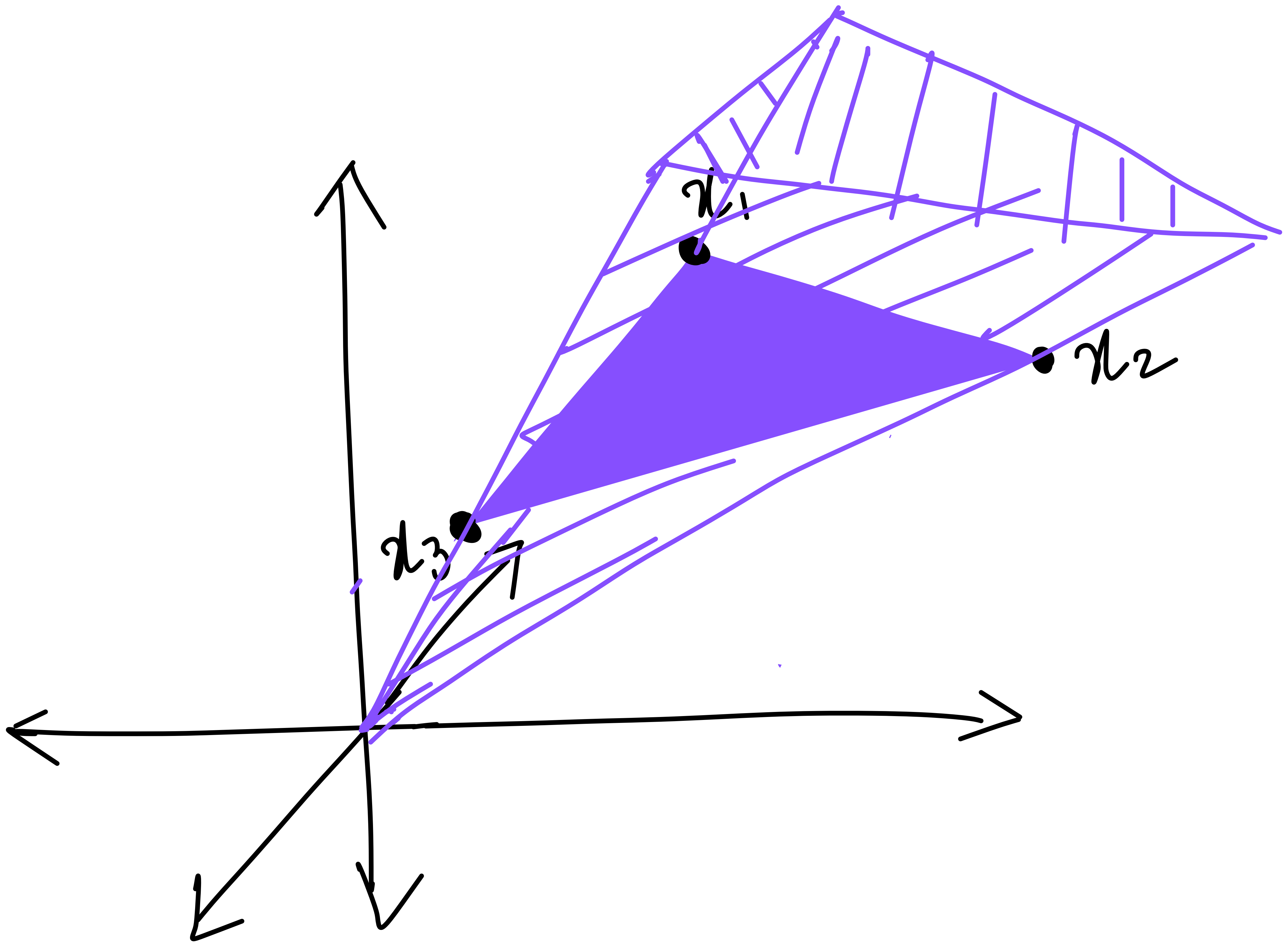
Cone

- A collection of points $x_1, \cdots, x_n$ and $\theta_1, \theta_2, \cdots, \theta_n \in \mathbb{R}^+$ induces the cone $\theta_1 x_1 + \cdots + \theta_n x_n$
- The coefficients are positive unlike for an affine set, and don't sum to 1
- E.g. Any $x_1 \neq x_2 \in \mathbb{R}^2$ generates a cone as shown in the figure



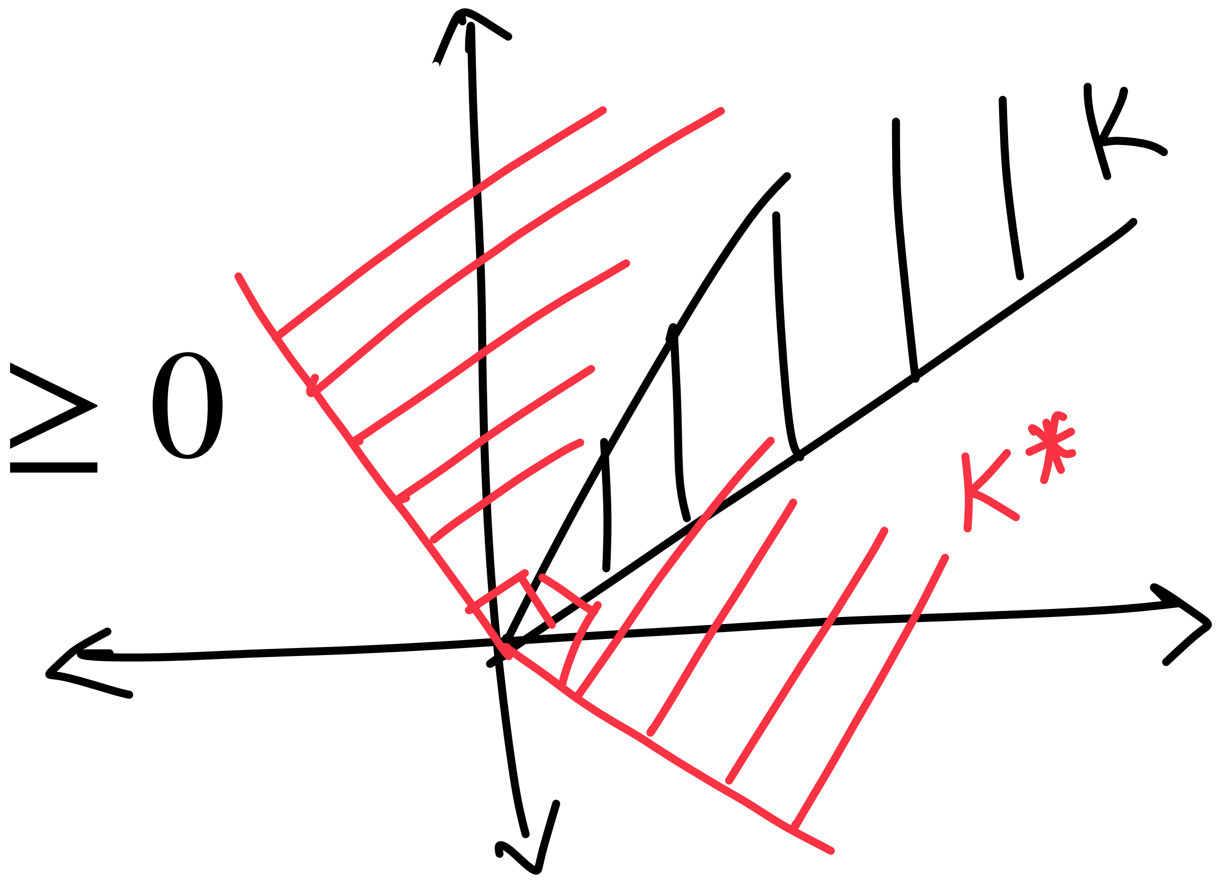
Cone

- E.g. II Non collinear $x_1, x_2, x_3 \in \mathbb{R}^3$ induces a pyramid cone



Dual Cone

- Dual cone of a cone K is defined as $K* = \{y \mid y^T x \geq 0, \forall x \in K\}$.
- $K*$ is also a cone (in fact it's a cone for any set K)
- For any $x_1, x_2 \in K*,$ let $z = \theta_1 x_1 + \theta_2 x_2$ for some $\theta_1, \theta_2 \geq 0$
- Clearly for any $y \in K,$ $y^T z = \theta_1 y^T x + \theta_2 y^T x \geq 0$
- Therefore $z \in K*$



Dual Cone

- Consider the dual of $K*,$ i.e. $(K*)* = \{y \,|\, y^T x \geq 0 \forall x \in K*\}$
- Clearly $K \subseteq (K*)*$ since any $y \in K$ has a positive inner product with any $x \in K*$ by definition of $K*$
- In fact for a closed convex K, $(K*)* = K$. Relies on the separation theorem

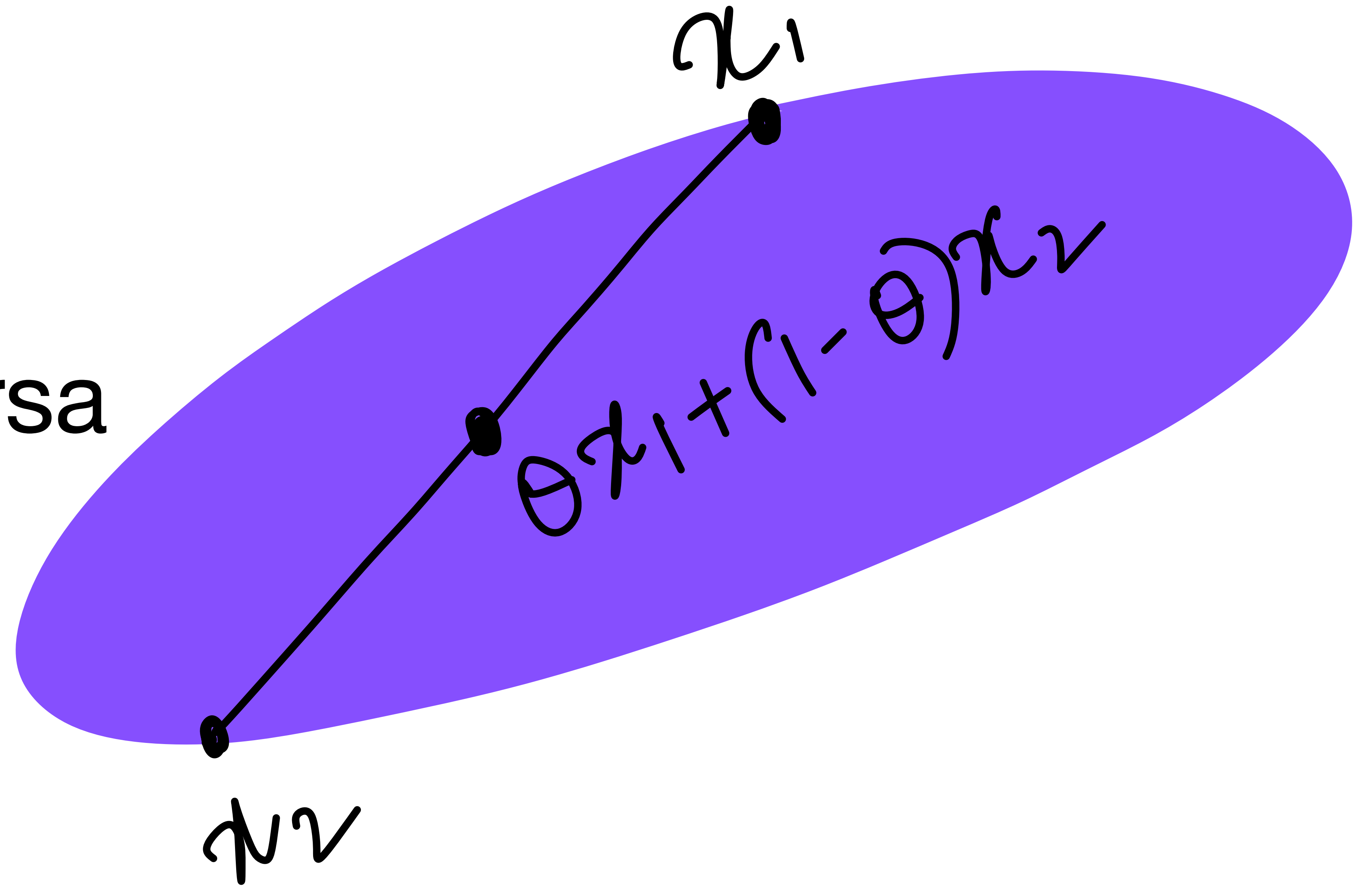
Convex set

- X is said to be convex if for any $x_1, x_2 \in X,$ and $0 \leq \theta \leq 1,\ \theta x_1 + (1-\theta)x_2 \in X$

- Can be more complicated to specify than an affine set or a cone stated in terms of 'boundary' points

- E.g. The purple region is a valid convex set X

- Affine set/cone $\implies$ convex set but not vice versa



Intersections of convex sets

- Consider convex sets $C_1, \cdots, C_n$ for some finite n. Then $C \triangleq C_1 \cap \cdots \cap C_n$ is also convex
- Consider any $x_1, x_2 \in C$. We need to show that $y = \theta x_1 + (1 - \theta)x_2 \in C$ for all $0 \leq \theta \leq 1$
- Clearly $x_1, x_2 \in C_i \forall i \in \{1,\cdots,n\}$, and so is $y \in C_i \forall i \in \{1,\cdots,n\}$
- Thus $y \in C$
- Shows that an optimization region with many convex constraints is convex

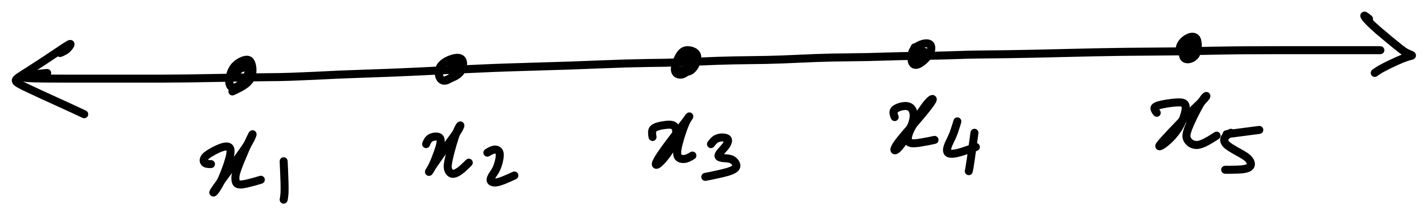
Hyperplanes

- A hyperplane is an affine set whose dimension is one less than that of its ambient space
- In general a set of the form $\{a + u : u \in U\}$ where a, U correspond to a point and a subspace
- E.g. the plane $x_1 + x_2 + 3x_3 = 5$ in $\mathbb{R}^3$. The original definition of 'hyperplane' is from the 3 dimensional case
- It's a convex set as it's affine (as affine $\implies$ convex)

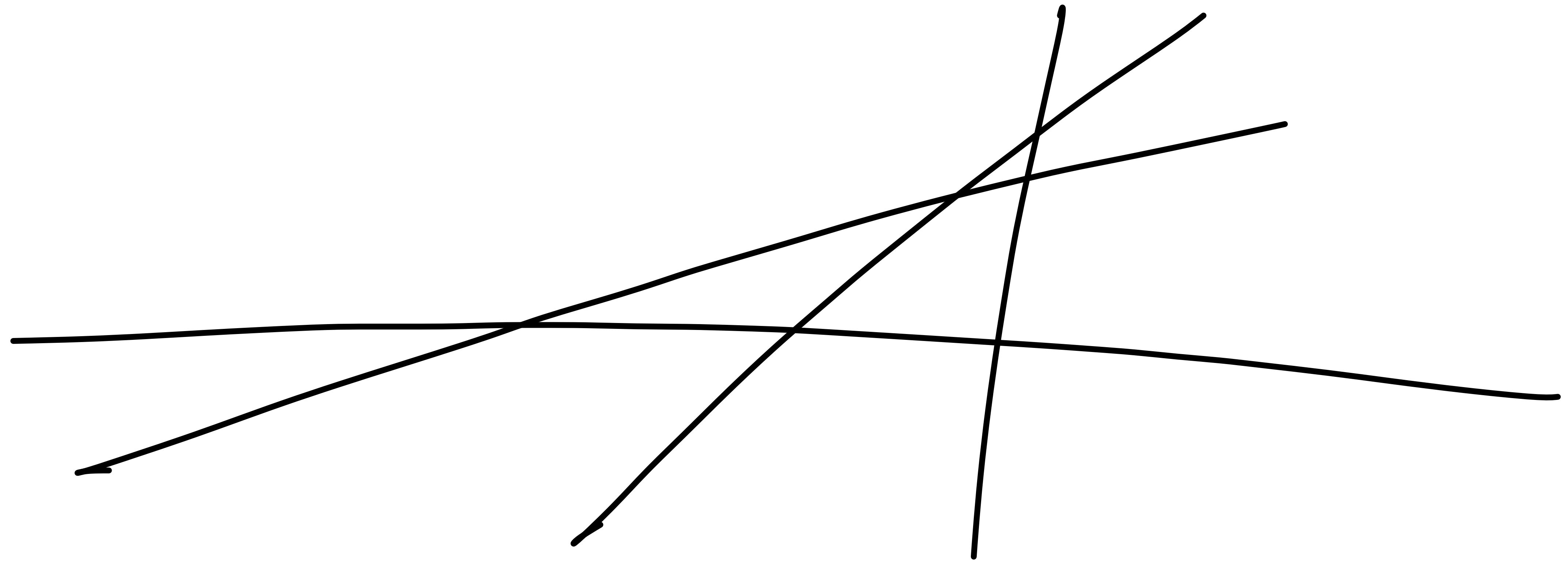
Hyperplanes and regions

- Consider $\mathbb{R}^n$. Given $a_i x = b_i$, for $i = 1,2,\ldots,p, \quad x \in \mathbb{R}^n$.
- We'd like to evaluate the maximum number of disjoint regions separated by the hyperplanes

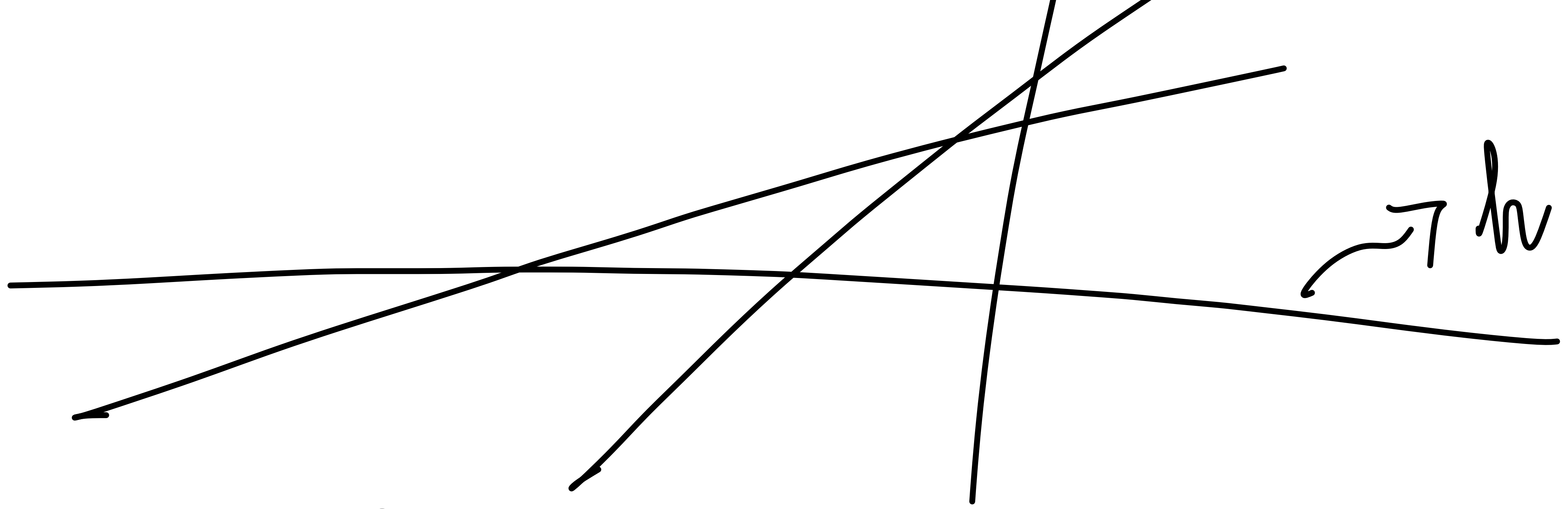
Illustrating $n = 1, 2$

- A hyperplane is a point for $n = 1$
- p points on a line segment. Clearly $p + 1$ regions
- E.g. 

& 6 regions
- Hyperplane in $\mathbb{R}^2$ consists of lines



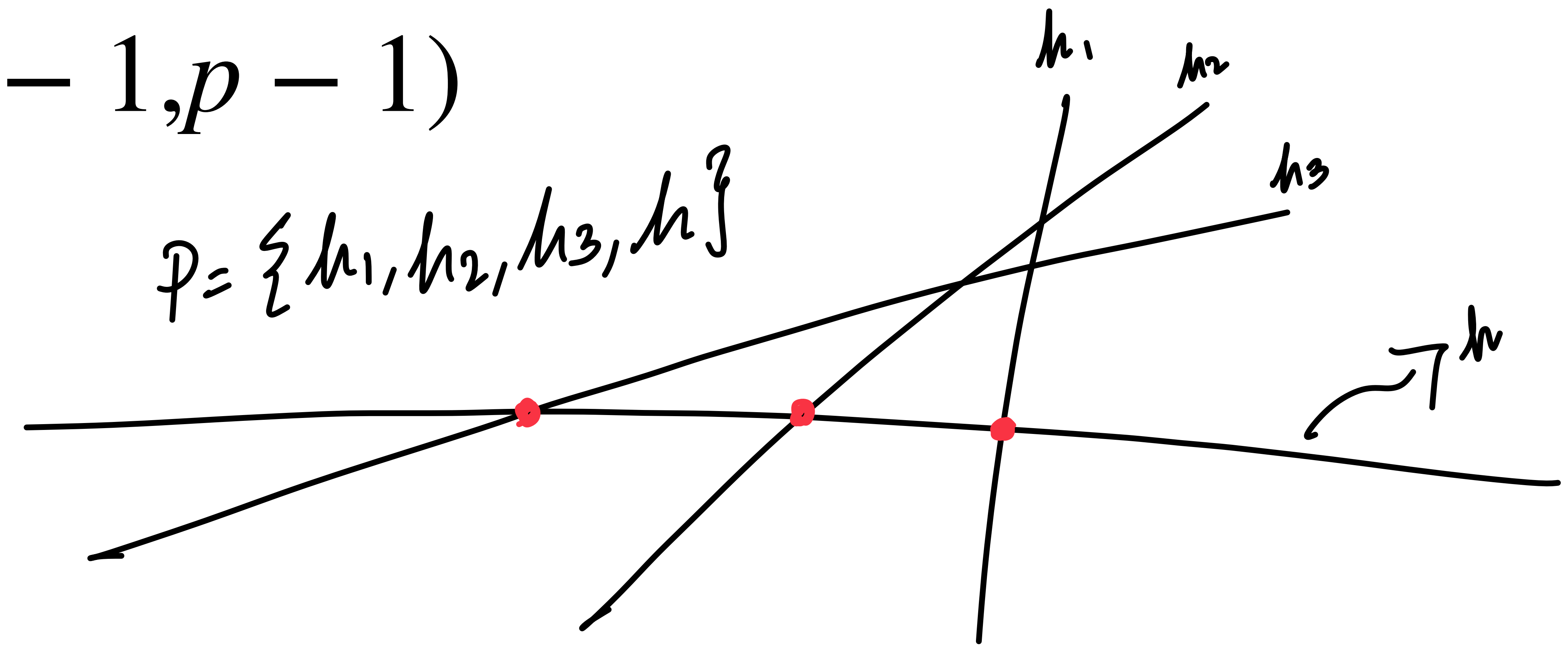
General n, p



- Let $M(n, p)$ denote the number of possible regions for general n, p
- From the discussion on $n = 1$ in the previous slide $M(1,p) = 1 + p = 1 + \binom{p}{1}$
- Key idea : We relate $M(n, p)$ with $M(n, p - 1)$ and $M(n - 1,p - 1)$.
- Let P denote hyperplane set. Without loss of generality assume all hyperplanes intersect.
- There exists a hyperplane $h \in P$ such that all hyperplanes in $P \backslash h$ intersect on one of its sides

Recurrence relation

- There exists a hyperplane $h \in P$ such that all hyperplanes in $P \backslash h$ intersect on one of its sides
- The number of regions on the side with intersections is $M(n, p-1)$
- Key: The number of regions on the other side is $M(n-1, p-1)$ as each region can be projected down to the hyperplane that is of dimension $n-1$. See figure
- Thus $M(n, p) = M(n, p-1) + M(n-1, p-1)$



General n, p

- We would like to solve for $M(n, p) = M(n, p-1) + M(n-1, p-1)$ using $M(1,p) = 1 + \binom{p}{1}$
- For $n = 2$, $M(2,p) = M(2,p-1) + M(1,p-1) = M(2,p-1) + \binom{p-1}{1} + 1$
- Can be proved via induction: $M(2,p) = 1 + \binom{p}{1} + \binom{p}{2}$
- For general n, p, similarly $M(n, p) = \sum_{i=0}^{n} \binom{p}{i}$