Midterm Review for CSE 203B

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Based on slides by Prof. Stephen Boyd

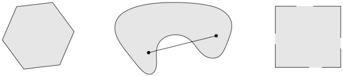
Logistics

- Released on course website & Piazza: `http://cseweb.ucsd.edu/classes/wi21/cse203B-a/`
- No time limit, submission on gradescope
- Released Saturday 2/19 10:00 am PST, due Tuesday 2/22 10:00 am PST
- 2 sections:
  - $\leq$ 10 True/False (with explanation)
  - $\leq$ 5 Derivations/simple proofs
  - One programming question
  - $\sim$ 70% based on homework questions

Overview

- Convex sets
- Convex functions
- Supporting hyperplanes
- Conjugate function
- Lagrangian Dual
- Logistics and other recommended topics

Convex sets: definition



- A set $S \subseteq \mathbb{R}^d$ is convex if the line segment between any two points in C lies in C: for any $x_1, x_2 \in C$ and $0 \leq \theta \leq 1$, $\theta x_1 (1 - \theta) x_2 \in C$

Convex sets: example

For any $x_1, x_2 \in C$ and $0 \leq \theta \leq 1$, $\theta x_1 (1 - \theta) x_2 \in C$

Example: the polytope

$\mathcal{K} = \{x | Ax \leq b\}$ for $x, b \in \mathbb{R}^d$, $A \in \mathbb{R}^{m \times n}$

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Example: the polytope

$P = \{x | Ax \leq b\}$ for $x, b \in \mathbb{R}^d$, $A \in \mathbb{R}^{m \times n}$

let $x_1, x_2 \in P$ and $0 \leq \lambda \leq 1$. Then

$A((1-\lambda)x_1 + \lambda x_2) = (1-\lambda)Ax_1 + \lambda A x_2 \leq (1-\lambda)b + \lambda b = b$

Or use a geometric argument: P is an intersection of m half-spaces.

Convex sets: example

For any $x_1, x_2 \in C$ and $0 \leq \theta \leq 1$, $\theta x_1 (1-\theta) x_2 \in C$

Example: dual cone

$$K = \{(x,t) : ||x||_1 \leq t\} \implies K^* = \{(x,t) : ||x||_\infty \leq t\}$$

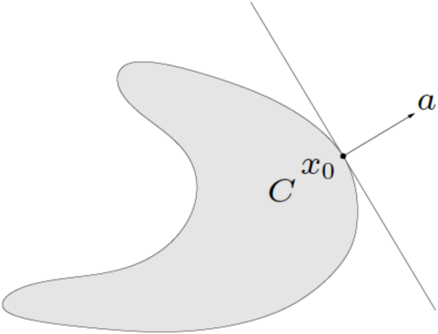
$$K^* = \{(x,t) : x^\top y + st \geq 0 : (x,t) \in K\}$$

Supporting Hyperplane Theorem

A supporting hyperplane to a set C is defined with respect to a boundary point x_0:

$$\{x | a^\top x = a^\top x_0\}$$

where $a \neq 0$ and $a^\top x \leq a^\top x_0$ for all $x \in C$.



Supporting hyperplane theorem: If C is convex, then there exists a supporting hyperplane at every boundary point of C.

Supporting Hyperplane: example

$t \geq$ $\boxed{\text{f(y)} \geq f(x) + \nabla f(x)^\top (y - x)}$ (first order condition)

$(y, t) \in \text{epi} f \implies \nabla f(x)^\top (y - x) + f(x) - t \leq 0$

$(\nabla f(x), -1)([y, t] - [x, f(x)]) \leq 0$

f'17 review: 2.1 $\{x | x_1^2 + 2x_2^2 \leq 9\}$ at $x_0 = [x_1, x_2] = [1, 2]$

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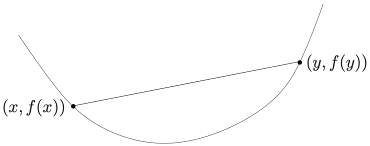
$\nabla f(x) = 2x_1 + 4x_2$, $(x, f(x)) = (x_1, x_2, f(x_0)) = (1, 2, 10)$

Normal vector at x_0: $(2, 8, -1)$.

Convex functions: definition

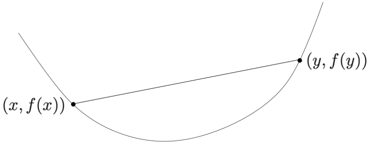


- A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom} f$ is a convex set and if for all $x, y \in \mathrm{dom} f$ and $0 \leq \theta \leq 1$

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y) \quad \text{Jensen's inequality}$$

- Concave functions: $-f$ is convex

Convex functions: definition



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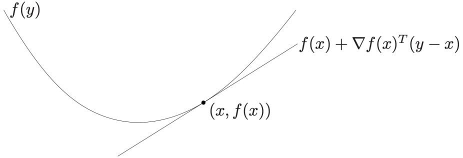
Convex functions: first order condition



- If f is differentiable ($\mathrm{dom} f$ is open, ∇f exists $\forall x \in \mathrm{dom} f$) then f is convex iff $\mathrm{dom} f$ is convex and for all $x, y \in \mathrm{dom\ f}$

$$f(y) \geq f(x) + \nabla f(x)^T(y - x)$$

Convex functions: second order condition

- Suppose f is twice-differentiable ($\mathrm{dom} f$ is open and its Hessian exists $\forall x \in \mathrm{dom} f$) then f is convex iff $\mathrm{dom} f$ $\mathrm{dom} f$ is convex and for all $x, y \in \mathrm{dom\ f}$

$$\nabla^2 f \succcurlyeq 0 \quad \text{(positive semidefinite)}$$

Convex functions: establishing convexity

By definition

- Show by definition or first-order condition
- For twice-differentiable functions, show $\nabla^2 f \succcurlyeq 0$

By convexity-preserving operations

- Nonnegative weighted sum
- Composition with affine function / composition with a convex + increasing function
- Pointwise maximum and supremum
- Composition
- Minimization

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Convex functions: examples

powers of absolute value

$f = |x|^p$ is convex with $p > 1$

Convex functions: examples

powers of absolute value

$f = |x|^p$ is convex with $p > 1$

Pf: Note that the composition of a convex and convex-increasing function is convex. Prove $|\cdot|$ is convex and x^p is convex and increasing.

Convex functions: examples

log-convex function

$g(x) = log(f(x))$, s.t. f convex.

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Convex functions: examples

quadratic form of inverse

$f : \mathbb{R}^n \times S^n \to \mathbb{R}$, $f(x, Y) = x^T Y^{-1} x$ is convex on $\mathbb{R}^n \times S^n_{++}$

Convex functions: examples

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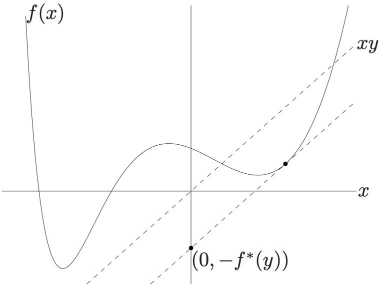
Show epigraph of f is a convex set. Express epigraph as an LMI and apply the definiteness conditions of the Schur Complement (appendix 5.5).

Conjugate function: definition



- Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the conjugate function

$$f^*(x) = \sup_{x \in \mathrm{dom} f} y^T x - f(x)$$

- $\mathrm{dom} f^*$ consists of $y \in \mathrm{dom} f$ such that $\sup_{y \in \mathrm{dom} f} y^T x - f(x)$ is bounded.
- $f^*(x)$ is convex even if $f(x)$ is not convex

Duality

Primal problem

$$\min f_0(x)$$
$$f_i(x) \leq 0$$
$$h_i(x) = 0$$

Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \end{aligned}$$

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\end{aligned}
$$

g is concave, can be unbounded for some $-\lambda$, ν.

Lower bound property

If $\lambda \geq 0$, then $g(\lambda, \nu) \leq p^*$.
proof: if $\bar{x}$ is feasible and $\lambda \geq 0$ then

$$
f_0(\bar{x}) \geq L(\bar{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(\bar{x}, \lambda, \nu) = g(\lambda, \nu)
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minimizing over all feasible $\bar{x}$ gives $p^* \geq g(\lambda, \nu)$.

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Duality example: Primal and Dual of a QCQP

$$\min_x x^\top A x + b^\top x$$
$$\text{s.t.} \quad x^\top x \leq c$$

- The feasible set is the ball $K = \{x | x^\top x \leq c\}$
- The Lagrange dual function of the primal problem is ?

Duality example: Primal and Dual of a QCQP

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- The Lagrange dual function of the primal problem is

$$\begin{aligned} g(\lambda) &= \inf_x (x^\top A x + b^\top x + \lambda(x^\top x - c)) \\ &= \inf x^\top (A + \lambda) x + b^\top x - \lambda c \\ &= (-1/4) b^\top (A + \lambda I)^{-1} b - (1/2)(A + \lambda I)^{-1} b - c\lambda^1 \end{aligned}$$

[1] Assume A invertible. Check $g(\lambda)$ is concave in λ

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Other

- Definitions and examples
- Classification of convex problems: LP, GP, SOCP, QCQP, etc.
- Characterization of PSD (convex) quadratic forms ($X \geq 0$)
  - $y^\top X y \geq 0$
  - All eigenvalues of $X \geq 0$
  - Unbounded below if $\lambda_{\min}(X) < 0$, otherwise 0.

Good Luck!