CSE 105
THEORY OF COMPUTATION

Winter 2022

https://cseweb.ucsd.edu/classes/wi22/cse105-a/
Today's learning goals

- Review what it means for a set to be closed under an operation.
- Define the regular operations on languages.
- Prove closure properties of the class of regular languages.

Sipser Ch 1.1, 1.2
Building DFA

Remember

States are our only (computer) memory.

Design and pick states with specific roles / tasks in mind.

"Have not seen any of desired pattern yet"
"Trap state"
Building DFA

New strategy

Express $L$ in terms of simpler languages – use them as building blocks.

Example

$L = \{ w \mid w \text{ does not contain the substring baba} \}$

$= \text{the complement of the set}$

$\{w \mid w \text{ contains the substring baba}\}$
Building DFA

DFA recognizing \( \{w | w \text{ contains the substring baba}\} \)

DFA recognizing \( \{w | w \text{ doesn't contain the substring baba}\} \)
Complementation

Theorem: If $A$ is a regular language over $\{0,1\}^*$, then so is its complement.

aka "the class of regular languages is closed under complementation"
Closure of \( \mathbb{Z} \) under addition.
- Set of even ints under multiplication.
- \( \{0\}^* \) under concatenation.

Which of these is true?

A. The set of odd integers is closed under addition.
X. The set of positive integers is closed under subtraction.
X. The set of rational numbers is closed under multiplication.
X. The set of real numbers is closed under division.
E. I don't know.
Complementation

**Theorem:** If $A$ is a regular language over $\{0,1\}^*$, then so is its complement

aka "the class of regular languages is closed under complementation"
Complementation

**Theorem**: If $A$ is a regular language over $\{0,1\}^*$, then so is its complement.

*aka "the class of regular languages is closed under complementation"

Proof: Let $A$ be a regular language. Then there is a DFA $M=(Q,\Sigma,\delta,q_0,F)$ such that $L(M) = A$. We want to build a DFA $M'$ whose language is $\overline{A}$. Define

$$M' = \text{?}$$

**Claim of Correctness** $L(M') = \overline{A}$

**Proof of claim**...
A is regular, accepted by DFA $M = (Q, \Sigma, \delta, q_0, F)$

define new DFA $M' = (Q, \Sigma, \delta, q_0, Q \setminus F)$

Claim: $M'$ accepts $A^c$.

Proof: because $M$ accepts $A$

$A = \{ w : M \text{ accepts } w \} = \{ w : \delta^*(q_0, w) \in F \}$

$A^c = \{ w : w \notin A \} = \{ w : \delta^*(q_0, w) \notin F \} = \{ w : \delta^*(q_0, w) \in Q \setminus F \}$

so $M'$ accepts $A^c$. 

$\delta^*(q, w) =$ state reached from $q$ after reading word $w$
Why closure proofs?

- General technique of proving a new language is regular
- Stretch the power of the model
- Puzzle!
The regular operations

For $A, B$ languages over same alphabet, define:

1. $A \cup B = \{ x | x \in A \text{ or } x \in B \}$
2. $A \circ B = \{ xy | x \in A \text{ and } y \in B \}$
3. $A^* = \{ x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A \}$

These are operations on sets!
Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation. If $A$ and $B$ are regular sets, then $A \cup B$ is also regular.

Proof:

What are we proving here?

- A. For any set $A$, if $A$ is regular then so is $A \cup A$.
- B. For any sets $A$ and $B$, if $A \cup B$ is regular, then so is $A$.
- C. For two DFAs $M_1$ and $M_2$, $M_1 \cup M_2$ is regular.
- D. None of the above.
- E. I don't know.
Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1, A_2$ be any two regular languages over $\Sigma$. WTS that $A_1 \cup A_2$ is regular.

Goal: build a DFA that recognizes $A_1 \cup A_2$. 
Union

Goal: build a DFA that recognizes $A_1 \cup A_2$.

Strategy: use DFAs that recognize each of $A_1, A_2$.

Accept if either (or both) accepts

**HOW?**
Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1$, $A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$. WTS that $A_1 \cup A_2$ is regular.

Define $M = (?, \Sigma, \delta, ?, ?)$
Union

Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1$, $A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$ and WTS that $A_1 \cup A_2$ is regular. Define $M = (Q_1 \times Q_2, \Sigma, \delta, ?, ?)$.
Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1, A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$ and WTS that $A_1 \cup A_2$ is regular. Define $M = (Q_1 \times Q_2, \Sigma, \delta, ?, ?)$

What should be the initial state of $M$?

A. $q_0$
B. $q_1$
C. $q_2$
D. $(q_1, q_2)$
E. I don't know.
Union

**Theorem**: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

**Proof**: Let $A_1, A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$, we want to show that $A_1 \cup A_2$ is regular.

Define $M = (Q_1 \times Q_2, \Sigma, \delta, ?, ?)$ where

- $r$ is a state in $M_1$, $s$ is a state in $M_2$, and $x$ is in $\Sigma$, then $\delta((r,s), x) =$

- A. $(r,s)$
- B. $\left( \delta(r,x), \delta(s,x) \right)$
- C. $\left( \delta_1(r,x), s \right)$
- D. $\left( \delta_1(r,x), \delta_2(s,x) \right)$
- E. I don't know.

$\Rightarrow$
Union

Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1, A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$ and WTS that $A_1 \cup A_2$ is regular.

Define $M = (Q_1 \times Q_2, \Sigma, \delta, ?, ?)$ where

$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$ for all $a \in \Sigma$.

The set of accepting states for $M$ is

- A. $F_1 \times F_2$
- B. $\{ (r, s) | r \text{ is in } F_1 \text{ and } s \text{ is in } F_2 \}$
- C. $\{ (r, s) | r \text{ is in } F_1 \text{ or } s \text{ is in } F_2 \}$
- D. $F_1 \cup F_2$
- E. I don't know.
Proof: Let $A_1, A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$.

WTS that $A_1 \cup A_2$ is regular. Define $M = (Q_1 \times Q_2, \Sigma, \delta, (q_1, q_2), \{(r, s) \in Q_1 \times Q_2 \mid r \in F_1 \text{ or } s \in F_2\})$ with $\delta((r, s), x) = (\delta_1(r, x), \delta_2(s, x))$ for each $(r, s)$ in $Q_1 \times Q_2$ and $x$ in $\Sigma$.

Why does $L(M) = A_1 \cup A_2$?
Intersection

• How would you prove that the class of regular languages is closed under intersection?

• Can you think of more than one proof strategy?

\[ A \cap B = \{ x \mid x \text{ in } A \text{ and } x \text{ in } B \} \]

\[ A \cap B = (A^c \cup B^c)^c \]

\[ \Rightarrow A^c, B^c \text{ regular} \]

\[ \Rightarrow A^c \cup B^c \text{ regular} \]

\[ \Rightarrow (A^c \cup B^c)^c \text{ regular} \]

\[ \Rightarrow A \cap B \text{ regular} \]
Payoff

\{ w \mid w \text{ contains neither the substrings } aba \text{ nor } baab \} 

Is this a regular set?
Payoff

\{ w \mid w \text{ contains neither the substrings } aba \text{ nor } baab \}

Is this a regular set?

A = \{ w \mid w \text{ contains } aba \text{ as a substring} \}
B = \{ w \mid w \text{ contains } baab \text{ as a substring} \}

\bar{A} \cap \bar{B} = \bar{A \cup B}
Sample closure proofs

• The class of regular languages over \{0,1\} is closed under the FlipBits operation, where
  \[ \text{FlipBits}(L) = \{ w \mid w \text{ is obtained from some } w' \text{ in } L \text{ by flipping each } 0 \text{ in } w \text{ to } 1, \text{ and each } 1 \text{ to } 0 \} \]

• The class of regular languages of \{a,b,z\} is closed under the DeleteWordsWithZ operation, where
  \[ \text{DeleteWordsWithZ}(L) = \{ w \mid w \text{ is in } L \text{ and } w \text{ doesn't contain } z \} \]
General proof structure/strategy

**Theorem:** For any $L$ over $\Sigma$, if $L$ is regular then [ the result of some operation on $L$ ] is also regular.

**Proof:**

*Given* name variables for sets, DFAs assumed to exist.

*WTS* state goal and outline plan.

*Construction* using objects previously defined + new tools working towards goal. Give formal definition and explain.

*Correctness* prove that construction works.

*Conclusion* recap what you've proved.
The regular operations

For A, B languages over same alphabet, define:

1. \( A \cup B = \{ x | x \in A \text{ or } x \in B \} \)
2. \( A \circ B = \{ xy | x \in A \text{ and } y \in B \} \)
3. \( A^* = \{ x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A \} \)

How can we prove that the concatenation of two regular languages is a regular language?