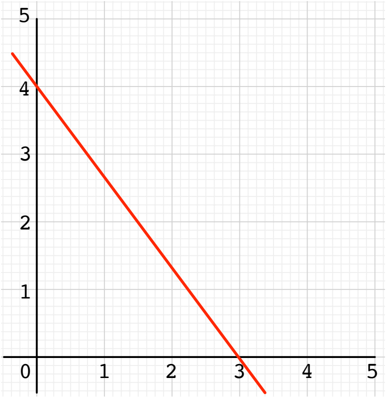
A simple linear classifier

CSE 250B

Linear decision boundary for classification: example



- What is the formula for this boundary?
- What label would we predict for a new point x?

Linear classifiers

Binary classification: data $x \in \mathbb{R}^d$ and labels $y \in \{-1, +1\}$

- Linear classifier:
  - Parameters: $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$
  - Decision boundary $w \cdot x + b = 0$
  - On point x, predict label $\text{sign}(w \cdot x + b)$
- If the true label on point x is y:
  - Classifier correct if $y(w \cdot x + b) > 0$

A loss function for classification

What is the **loss** of the linear classifier w, b on a point (x, y)?

One idea for a loss function:

- If $y(w \cdot x + b) > 0$: correct, no loss
- If $y(w \cdot x + b) < 0$: loss $= -y(w \cdot x + b)$

A simple learning algorithm

Fit a linear classifier w, b to the training set using **stochastic gradient descent**.

- Update w, b based on just one data point (x, y) at a time
- If $y(w \cdot x + b) > 0$: zero loss, no update
- If $y(w \cdot x + b) \leq 0$: loss is $-y(w \cdot x + b)$

A simple learning algorithm

Fit a linear classifier w, b to the training set using **stochastic gradient descent**.

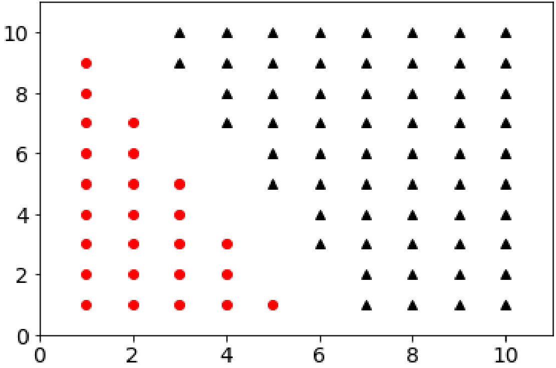
- Update w, b based on just one data point (x, y) at a time
- If $y(w \cdot x + b) > 0$: zero loss, no update
- If $y(w \cdot x + b) \leq 0$: loss is $-y(w \cdot x + b)$

The Perceptron algorithm

- Initialize $w = 0$ and $b = 0$
- Keep cycling through the training data (x, y):
  - If $y(w \cdot x + b) \leq 0$ (i.e. point misclassified):
    - $w = w + yx$
    - $b = b + y$

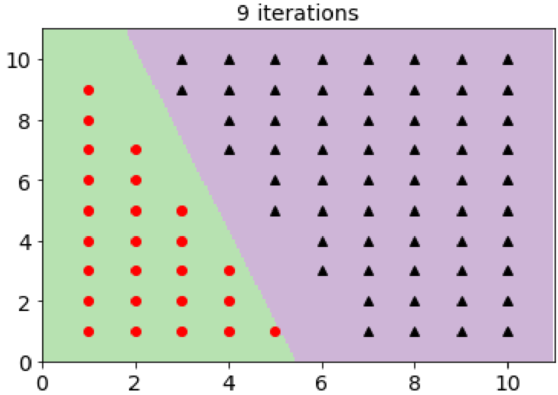
The Perceptron in action

85 data points, linearly separable.



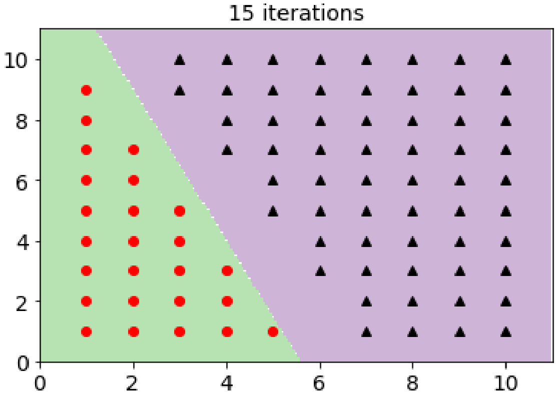
The Perceptron in action

85 data points, linearly separable.



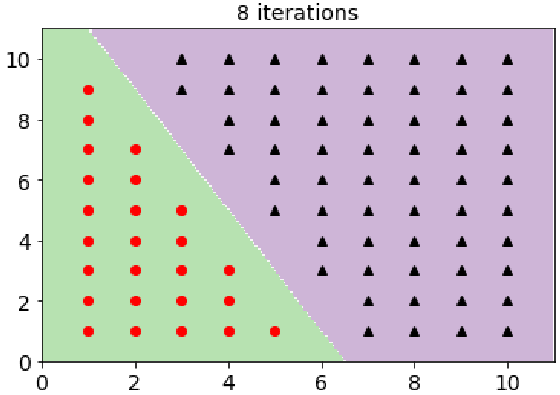
The Perceptron in action

85 data points, linearly separable.



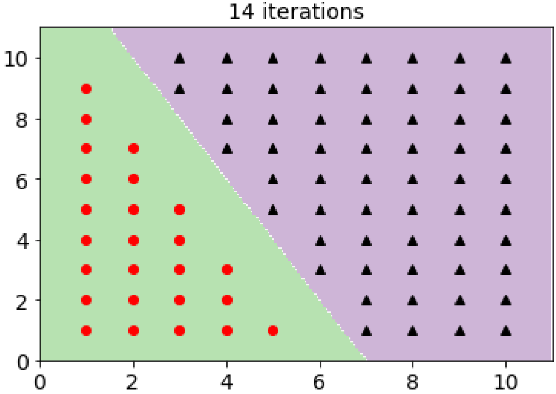
The Perceptron in action

85 data points, linearly separable.



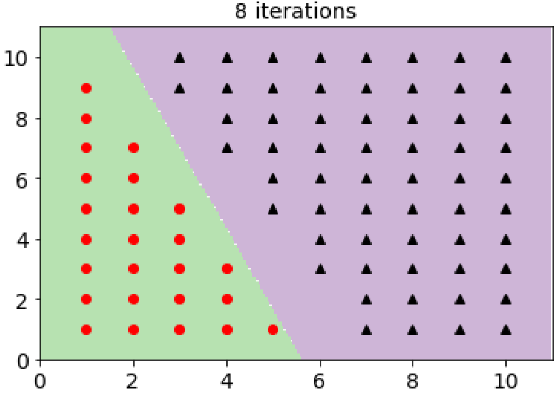
The Perceptron in action

85 data points, linearly separable.



The Perceptron in action

85 data points, linearly separable.



Perceptron: convergence

Theorem: Let $R = \max \|x^{(i)}\|$. Suppose there is a unit vector w^* and some (margin) $\gamma > 0$ such that

$$y^{(i)}(w^* \cdot x^{(i)}) \geq \gamma \ \text{ for all } i.$$

Then the Perceptron algorithm converges within R^2/γ^2 updates.

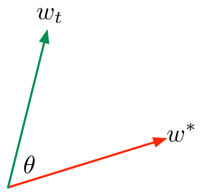
Perceptron: convergence

Theorem: Let $R = \max \|x^{(i)}\|$. Suppose there is a unit vector w^* and some (margin) $\gamma > 0$ such that

$$y^{(i)}(w^* \cdot x^{(i)}) \geq \gamma \text{ for all } i.$$

Then the Perceptron algorithm converges within R^2/γ^2 updates.

Proof idea. Let w_t be the classifier after t updates.



Track angle between w_t and w^*:

$$\cos(\angle(w_t, w^*)) = \frac{w_t \cdot w^*}{\|w\|}.$$

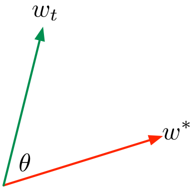
Perceptron: convergence

Theorem: Let $R = \max \|x^{(i)}\|$. Suppose there is a unit vector w^* and some (margin) $\gamma > 0$ such that

$$y^{(i)}(w^* \cdot x^{(i)}) \geq \gamma \text{ for all } i.$$

Then the Perceptron algorithm converges within R^2/γ^2 updates.

Proof idea. Let w_t be the classifier after t updates.



Track angle between w_t and w^*:

$$\cos(\angle(w_t, w^*)) = \frac{w_t \cdot w^*}{\|w\|}.$$

On each mistake, when w_t is updated to w_{t+1},

- $w_t \cdot w^*$ grows significantly.
- $\|w_t\|$ does not grow much.