CSE 250B Winter 2020 - Quiz 3

Feb 21, 2020 - 30 Minutes **Name:** **PID:**

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(1) This is a closed book, closed notes exam. Switch off your cell phone and do not communicate with anyone other than an exam proctor.
(2) Start writing when instructed. Stop writing when your time is up.
(3) Remember that your work is graded on the quality of your writing and explanation as well as the validity of the mathematics.

(1) In each of the following cases, state if $f(x)$ is a convex function. Justify your answer.

(a) (5 points) $f(x) = e^{-x^\top x}$.

$$\frac{\partial f}{\partial x_i} = e^{-x^T x}(-2x_i)$$
$$\frac{\partial f}{\partial x} = e^{-x^T x}(-2x)$$
$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \begin{cases} e^{-x^T x}(4x_i^2 - 2) & x_i = x_j \\ e^{-x^T x}(4x_i x_j) & x_i \neq x_j \end{cases}$$
$$\therefore H = \frac{\partial^2 f}{\partial x^2} = e^{-x^T x}(4x^T x - 2I)$$

For a vector of all ones z, $z^T H z < 0$ at $x = 0$. Therefore, H is not PSD.

(b) (5 Points) Suppose g and h are convex functions (not necessarily differentiable). $f(x) = g(x) + h(x)$.

Since g and h are convex functions, for any x and y

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$$
$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y)$$

For any x and $y \in D_f$,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= g(\lambda x + (1-\lambda)y) + h(\lambda x + (1-\lambda)y) \\ &\leq \lambda g(x) + (1-\lambda)g(y) + \lambda h(x) + (1-\lambda)h(y) \\ &= \lambda g(x) + \lambda h(x) + (1-\lambda)g(y) + (1-\lambda)h(y) \\ &= \lambda f(x) + (1-\lambda)f(y) \end{aligned}$$

(2) (5 points) Suppose A and B are convex sets. Is the following set convex? Justify your answer.

$$C = \{x + y | x \in A, y \in B\}$$

$$\begin{aligned}
\text{Let} z_1 = x + y \in C & \quad \implies x \in A, y \in B \\
\text{Let} z_2 = a + b \in C & \quad \implies a \in A, b \in B \\
\text{For any } \lambda \in [0,1], & \\
\lambda x + (1-\lambda)a \in A & \quad \because x \in A, a \in A \\
\lambda y + (1-\lambda)b \in B & \quad \because y \in B, b \in B \\
\therefore \lambda x + (1-\lambda)a + \lambda y + (1-\lambda)b \in C & \quad \because \lambda x + (1-\lambda)a \in A, \lambda y + (1-\lambda)b \in B \\
\implies \lambda(x+y) + (1-\lambda)(a+b) \in C & \\
\therefore \lambda z_1 + (1-\lambda) z_2 \in C & \\
\therefore \text{ C is convex.} &
\end{aligned}$$