CSE 151A Winter 2021 - Take Home Exam 1

1 hour **Name:** **PID:**

(1) This is an open book, open notes exam. You are free to consult any text book or notes. **You are not allowed to consult with any other person.**
(2) If you need any clarification, please post a private message to the instructors on Piazza.
(3) Remember that your work is graded on the *clarity* of your writing and explanation as well as the validity of what you write.
(4) This is a one-hour exam.

(1) In class we defined positive semidefinite (PSD) matrices. A $d \times d$ matrix A is said to be positive definite (PD) if for all $d \times 1$ vectors x that are not the all-zeros vector, we have that $x^\top Ax > 0$. Observe that a positive definite matrix is also positive semi-definite, but a positive semi-definite matrix may not be positive definite.

For each of the following matrices, state if they are (a) positive definite (b) positive semi-definite but not positive definite (c) not positive semi-definite. Justify your answer with a short proof.

(a) (3 points) Let u be a $d \times 1$ unit vector – that is, $\|u\| = 1$. $A = uu^\top$.

For any nonzero vector x, we have

$$x^\top Ax = x^\top uu^\top x = \left(u^\top x\right)^\top u^\top x = \langle u, x\rangle^2 \geq 0\,.$$

Now, to show that A is not positive definite, we can pick a unit vector v that is perpendicular to u. Then, we have

$$v^\top Av = \langle u, v\rangle^2 = 0\,.$$

Thus, A is positive semi-definite, but not positive definite.

(b) (3 points) $B = 2I_d$. (Recall I_d is the $d \times d$ identity matrix.)

For any nonzero vector x, we have

$$x^\top Bx = x^\top 2I_d x = 2x^\top x = 2\|x\|^2 > 0\,.$$

Note that the equality does not hold as x is nonzero. Thus, B is positive definite.

(c) (4 points) Let u be a $d \times 1$ unit vector – that is, $\|u\| = 1$. $C = uu^\top - \frac{1}{2}I_d$.

Let us pick a unit vector v that is perpendicular to u. Then, we have

$$v^\top Cv = v^\top \left(uu^\top - \frac{1}{2}I_d\right) v = v^\top uu^\top v - \frac{1}{2}v^\top v = \mathbf{0} \cdot \mathbf{0} - \frac{1}{2}\|v\|^2 = -\frac{1}{2}\,.$$

This violates the condition of positive semi-definite. Thus, C is not positive semi-definite.

(2) (5 points) Let x be a $d \times 1$ vector, let $z_1, \ldots, z_n$ be n vectors, where each z_i is $d \times 1$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be the following multivariate function of x:

$$f(x) = \sum_{i=1}^{n} e^{x^\top z_i}$$

Write down the gradient of f with respect to x. Write down an algorithm that computes this gradient in $O(nd)$ time when given the vectors $z_1, \ldots, z_n$.

First, we note that

$$f(x) = \sum_{i=1}^{n} e^{x^\top z_i} = \sum_{i=1}^{n} e^{z_{i1}x_1 + z_{i2}x_2 + \cdots + z_{id}x_d} = \sum_{i=1}^{n} e^{z_{i1}x_1} e^{z_{i2}x_2} \ldots e^{z_{id}x_d} .$$

Then, we can compute the partial derivative with respect to x_j as follows.

$$\begin{aligned} \frac{\partial f}{\partial x_j} &= \sum_{i=1}^{n} e^{z_{i1}x_1} e^{z_{i2}x_2} \cdots (z_{ij} e^{z_{ij}x_j}) \cdots e^{z_{id}x_d} && \text{(by product rule)} \\ &= \sum_{i=1}^{n} e^{z_{i1}x_1 + z_{i2}x_2 + \cdots + z_{id}x_d} z_{ij} \\ &= \sum_{i=1}^{n} e^{x^\top z_i} z_{ij} . \end{aligned}$$

Thus, the gradient of f is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \sum_{i=1}^{n} e^{x^\top z_i} z_{i1} \\ \sum_{i=1}^{n} e^{x^\top z_i} z_{i2} \\ \vdots \\ \sum_{i=1}^{n} e^{x^\top z_i} z_{id} \end{bmatrix} = \sum_{i=1}^{n} e^{x^\top z_i} z_i .$$

This can be computed in $O(nd)$ time as follows.

Algorithm 1: Compute the gradient

Input: A $d \times 1$ vector x and n $d \times 1$ vectors $z_1, z_2, \ldots, z_n$

Output: A $d \times 1$ vector representing the gradient $\partial f / \partial x$

Let g be an array of size d, initialized to zeros

for $i = 1$ **to** n **do**

  $y \leftarrow x^\top z_i$ `// This can be done in` $O(d)$ `time`

  for $j = 1$ **to** d **do**

    $g[j] \leftarrow g[j] + e^y z_{ij}$ `// This can be done in` $O(1)$ `time`

  end

end

return g

(3) (5 points) Given a $d \times d$ matrix A, its Frobenius norm is defined as:

$$\|A\|_F = \sqrt{\sum_{i=1}^{d} \sum_{j=1}^{d} A_{ij}^2}$$

Now suppose we are given two $d \times 1$ vectors x and y. State whether the following statement is true or false. Justify your answer.

$$x^\top y \leq \|xy^\top\|_F$$

Let $A = xy^\top$. We have

$$\begin{aligned}
\left(x^\top y\right)^2 &= \left(\sum_{i=1}^{d} x_i y_i\right)^2 \\
&\leq \left(\sum_{i=1}^{d} x_i^2\right)\left(\sum_{i=1}^{d} y_i^2\right) && \text{(by Cauchy-Schwarz inequality)} \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} x_i^2 y_j^2 \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} A_{ij}^2 && \text{(by definition, } A_{ij} = x_i y_j\text{)} \\
&= \|xy^\top\|_F^2
\end{aligned}$$

Since $\|xy^\top\|_F$ is always nonnegative, we can conclude that

$$x^\top y \leq \|xy^\top\|_F\,.$$

This problem is similar to Question 4 in Homework1, where we showed that $\|xy^\top\|_F = \|x\|\|y\|$.