CSE 151A Winter 2021 - Take Home Exam 0

1 hour **Name:** **PID:**

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(1) This is an open book, open notes exam. You are free to consult any text book or notes. **You are not allowed to consult with any other person.**

(2) If you need any clarification, please post a private message to the instructors on Piazza.

(3) Remember that your work is graded on the *clarity* of your writing and explanation as well as the validity of what you write.

(4) This is a one-hour exam.

(1) Let X and Y be random variables with the following joint distribution.

	$X = 1$	$X = 2$	$X = 3$
$Y = 0$	1/4	1/4	1/4
$Y = 1$	1/12	0	1/6

Now answer the following questions.

(a) (4 points) What are the marginal distributions of X and Y?

To calculate the marginal distribution of X, we use the formula: $\Pr(X = x) = \sum_{y=0,1} \Pr(X = x, Y = y)$. This gives us the marginal distribution:

$$\Pr(X = 1) = 1/3, \Pr(X = 2) = 1/4, \Pr(X = 3) = 5/12$$

Similarly to calculate the marginal distribution of Y, we use: $\Pr(Y = y) = \sum_{x=1,2,3} \Pr(X = x, Y = y)$. This gives us the distribution:

$$\Pr(Y = 0) = 3/4, \Pr(Y = 1) = 1/4$$

(b) (4 points) What are the conditional expectations $\mathbb{E}[X|Y = 1]$ and $\mathbb{E}[Y|X = 1]$?

We first calculate the conditional distributions of $X|Y = 1$ and $Y|X = 1$ using the Bayes Rule: $\Pr(X = x|Y = y) = \frac{\Pr(X=x,Y=y)}{\Pr(Y=y)}$. The conditional distribution of $X|Y = 1$ is:

$$\Pr(X = 1|Y = 1) = 1/3, \Pr(X = 2|Y = 1) = 0, \Pr(X = 3|Y = 1) = 2/3$$

The conditional distribution of $Y|X = 1$ can be similarly calculated:

$$\Pr(Y = 0|X = 1) = 3/4, \Pr(Y = 1|X = 1) = 1/4$$

(c) (2 points) Are X and Y independent? Justify your answer.

X and Y are not independent. To see this, observe that $\Pr(X = 2, Y = 1) = 0$ while $\Pr(X = 2) \times \Pr(Y = 1) = 1/4 \times 1/4 = 1/16$.

(2) (5 points) Let $a_1, \ldots, a_k$ be k real numbers. Consider the following function:

$$f(x) = \sum_{i=1}^{k} \log(1 + e^{a_i x})$$

Write down the derivative of $f(x)$. Use this derivative to write down a condition on the numbers $a_1, \ldots, a_k$ that ensures that $f(x)$ is strictly increasing at $x = 0$.

The derivative is:

$$f'(x) = \sum_{i=1}^{k} \frac{1}{1 + e^{a_i x}} \cdot e^{a_i x} \cdot a_i = \sum_{i=1}^{k} \frac{a_i}{1 + e^{-a_i x}}$$

$f(x)$ is strictly increasing at $x = 0$ if $f'(0) > 0$. In other words:

$$\sum_{i=1}^{k} \frac{a_i}{2} > 0$$

which is the same as $\sum_{i=1}^{k} a_i > 0$.

(3) (5 points) Let $v_1 = [3, 5, 1]$, $v_2 = [2, 1, 3]$ and $v_3 = [12, 13, 11]$. Are v_1, v_2 and v_3 linearly independent? Justify your answer. To check if they are linearly independent, we try Gaussian elimination of the matrix:

$$\begin{bmatrix} 3 & 5 & 1 \\ 2 & 1 & 3 \\ 12 & 13 & 11 \end{bmatrix}$$

To eliminate Row 1, we do: $R2 \leftarrow R2 - \frac{2}{3} \times R1$ and $R3 \leftarrow R3 - 4R1$. This gives us the matrix:

$$\begin{bmatrix} 3 & 5 & 1 \\ 0 & -\frac{7}{3} & \frac{7}{3} \\ 0 & -7 & 7 \end{bmatrix}$$

Now, we can eliminate Row 2 by doing the operation: $R3 \leftarrow R3 - 2R2$ to get:

$$\begin{bmatrix} 3 & 5 & 1 \\ 0 & -\frac{7}{3} & \frac{7}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Since this matrix has an all-zero row, it has rank 2, and so does the matrix $\begin{bmatrix} 3 & 5 & 1 \\ 2 & 1 & 3 \\ 12 & 13 & 11 \end{bmatrix}$. Thus the vectors v_1, v_2 and v_3 are not linearly independent.