Linear algebra and geometric transformations in 2D

Computer Graphics

CSE 167

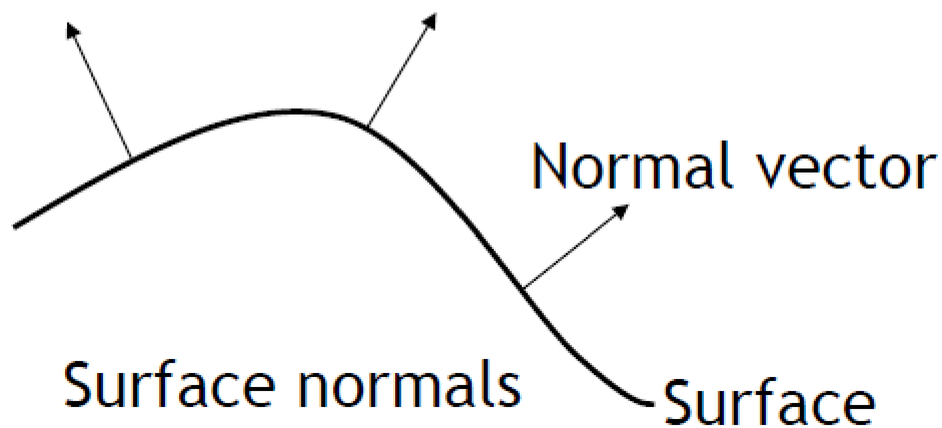
Lecture 2

CSE 167: Computer Graphics

- Linear algebra
  - Vectors
  - Matrices
- Points as vectors
- Geometric transformations in 2D
  - Homogeneous coordinates

Vectors

- Represent magnitude and direction in multiple dimensions
- Examples
  - Translation of a point
  - Surface normal vectors (vectors orthogonal to surface)



Vectors and arithmetic

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Vectors are column vectors

- Addition and subtraction

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \qquad \mathbf{a} - \mathbf{b} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{bmatrix}$$

Vectors must be the same length

Vectors and arithmetic

- Negate vector

$$-\mathbf{a} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}$$

- Vector-scalar multiply

$$s\mathbf{a} = \mathbf{a}s = \begin{bmatrix} sa_1 \\ sa_2 \\ \vdots \\ sa_n \end{bmatrix} \text{ where } s \text{ is a scalar}$$

Vectors and arithmetic

Examples using 3-vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} \quad \mathbf{a} - \mathbf{b} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}$$

$$-\mathbf{a} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} \qquad s\mathbf{a} = \begin{bmatrix} sa_1 \\ sa_2 \\ sa_3 \end{bmatrix} \text{ where } s \text{ is a scalar}$$

Vector transpose

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Column vector

$$\mathbf{a}^\top = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

Row vector

Example using 3-vector

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\mathbf{a}^\top = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

Magnitude of a vector

- The magnitude of a vector is its norm

$$\|\mathbf{v}\| = \sqrt{\sum_i v_i^2}$$

Example using 3-vector

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Unit vector

- A vector of magnitude 1 is called a unit vector
- A vector can be unitized by dividing by its norm

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

- The norm of a unit vector is 1

$$\|\hat{\mathbf{v}}\| = 1$$

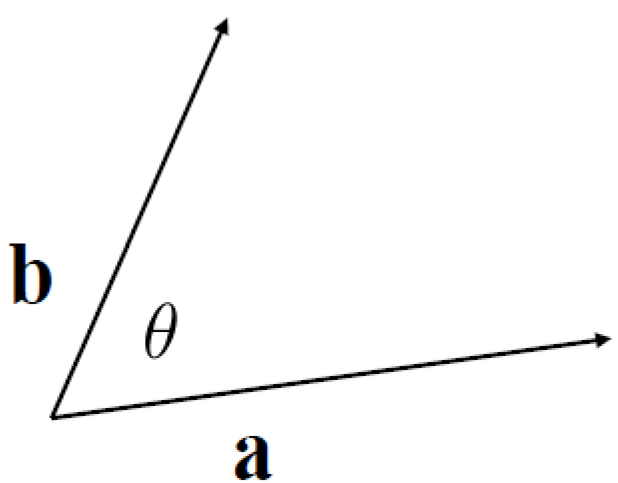
Dot product of two vectors

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\top \mathbf{b} = \sum_i a_i b_i$$

Example using 3-vector

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\top \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Angle between two vectors



$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\top \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

$$\cos \theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

$$\theta = \cos^{-1} \left(\frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right)$$

$$\theta = \cos^{-1} \left(\hat{\mathbf{a}}^\top \hat{\mathbf{b}} \right)$$

Magnitude of a vector revisited

- The magnitude of a vector is its norm

$$\|\mathbf{v}\| = \sqrt{\sum_i v_i^2}$$

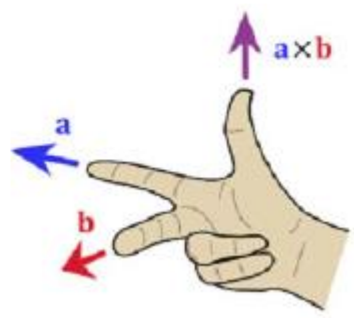
$$\|\mathbf{v}\| = \sqrt{\mathbf{v}^\top \mathbf{v}}$$ Using dot product

$$\|\mathbf{v}\|^2 = \mathbf{v}^\top \mathbf{v}$$

Cross product of two 3-vectors

- The cross product of two 3-vectors **a** and **b** results in another 3-vector that is orthogonal (using right hand rule) to the two vectors

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$



Cross product of two 3-vectors

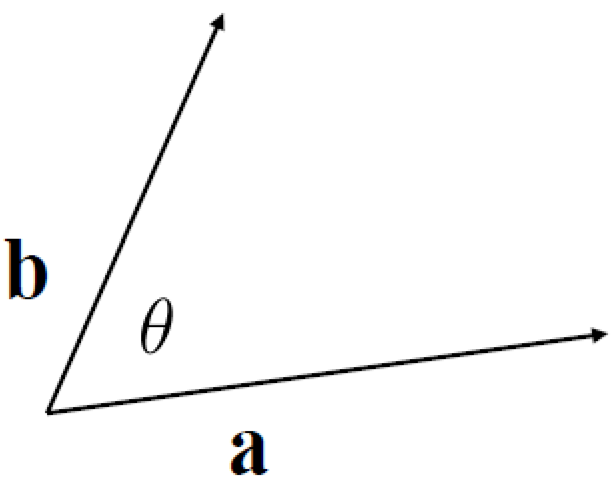
- Angle between two vectors

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$$

$$\sin\theta = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

$$\theta = \sin^{-1}\left(\frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)$$

$$\theta = \sin^{-1}\left(\|\hat{\mathbf{a}} \times \hat{\mathbf{b}}\|\right)$$



Matrices

- 2D array of numbers

$$\mathtt{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

m rows and *n* columns

Matrix addition and subtraction

- Matrices must be the same size

$$\mathtt{A} + \mathtt{B} = \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots & a_{1,n} + b_{1,n} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \cdots & a_{2,n} + b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & a_{m,2} + b_{m,2} & \cdots & a_{m,n} + b_{m,n} \end{bmatrix}$$

$$\mathtt{A} - \mathtt{B} = \begin{bmatrix} a_{1,1} - b_{1,1} & a_{1,2} - b_{1,2} & \cdots & a_{1,n} - b_{1,n} \\ a_{2,1} - b_{2,1} & a_{2,2} - b_{2,2} & \cdots & a_{2,n} - b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} - b_{m,1} & a_{m,2} - b_{m,2} & \cdots & a_{m,n} - b_{m,n} \end{bmatrix}$$

Matrix-scalar multiplication

$$s\mathtt{A} = \mathtt{A}s = \begin{bmatrix} sa_{1,1} & sa_{1,2} & \cdots & sa_{1,n} \\ sa_{2,1} & sa_{2,2} & \cdots & sa_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ sa_{m,1} & sa_{m,2} & \cdots & sa_{m,n} \end{bmatrix}$$

Matrix-vector multiplication

$$\mathtt{A}\mathbf{x} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix of *m* rows and *n* columns
times
vector of length *n*

Number of matrix columns must equal vector length

$$\mathtt{A}\mathbf{x} = \begin{bmatrix} \mathbf{a}^{1\top} \\ \mathbf{a}^{2\top} \\ \vdots \\ \mathbf{a}^{m\top} \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{a}^{i\top} \text{ is the } i\text{-th row of } \mathtt{A}$$

$$\mathtt{A}\mathbf{x} = \begin{bmatrix} \mathbf{a}^{1\top}\mathbf{x} \\ \mathbf{a}^{2\top}\mathbf{x} \\ \vdots \\ \mathbf{a}^{m\top}\mathbf{x} \end{bmatrix}$$

Result is vector of length *m* (i.e., number of rows in matrix)

Matrix-vector multiplication

Example using
3x3 matrix
and
3-vector

$$\mathtt{Ax} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathtt{Ax} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 \end{bmatrix}$$

Cross product of two 3-vectors revisited

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Using matrix-vector multiply

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}$$

$$\text{where } [\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Matrix-matrix multiplication

$$\mathtt{AB} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,p} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,p} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p,1} & b_{p,2} & \cdots & b_{p,n} \end{bmatrix}$$

m-by-*p* matrix times *p*-by-*n* matrix

Number of left matrix columns must equal number of right matrix rows

$$\mathtt{AB} = \begin{bmatrix} \mathbf{a}^{1\top} \\ \mathbf{a}^{2\top} \\ \vdots \\ \mathbf{a}^{m\top} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}, \text{ where } \mathbf{a}^{i\top} \text{ is the } i\text{-th row of } \mathtt{A} \text{ and } \mathbf{b}_j \text{ is the } j\text{-th column of } \mathtt{B}$$

$$\mathtt{AB} = \begin{bmatrix} \mathbf{a}^{1\top}\mathbf{b}_1 & \mathbf{a}^{1\top}\mathbf{b}_2 & \cdots & \mathbf{a}^{1\top}\mathbf{b}_n \\ \mathbf{a}^{2\top}\mathbf{b}_1 & \mathbf{a}^{2\top}\mathbf{b}_2 & \cdots & \mathbf{a}^{2\top}\mathbf{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}^{m\top}\mathbf{b}_1 & \mathbf{a}^{m\top}\mathbf{b}_2 & \cdots & \mathbf{a}^{m\top}\mathbf{b}_n \end{bmatrix}$$

Result is *m*-by-*n* matrix

Matrix transpose

$$\mathtt{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

m rows and *n* columns

$$\mathtt{A}^\top = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{bmatrix}$$

n rows and *m* columns

Flipped over the diagonal

- The transpose of a product of matrices is the transpose of each matrix multiplied in reverse order

$$(\mathtt{ABC})^\top = \mathtt{C}^\top \mathtt{B}^\top \mathtt{A}^\top$$

Example using three matrices

The identity matrix

$$\mathtt{I}_1 = 1, \mathtt{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathtt{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots, \mathtt{I}_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\mathtt{AI} = \mathtt{IA} = \mathtt{A}$ for any square matrix $\mathtt{A}$

Matrix inverse

- The inverse of a square matrix **A** is a matrix $\mathbf{A}^{-1}$ such that

$$AA^{-1} = A^{-1}A = I$$

- A square matrix has an inverse if and only if its determinant is nonzero
- The inverse of a product of matrices is the inverse of each matrix multiplied in reverse order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Example using three matrices

Representing points using vectors

- 2D point

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- 3D point

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Geometric transformations in 2D

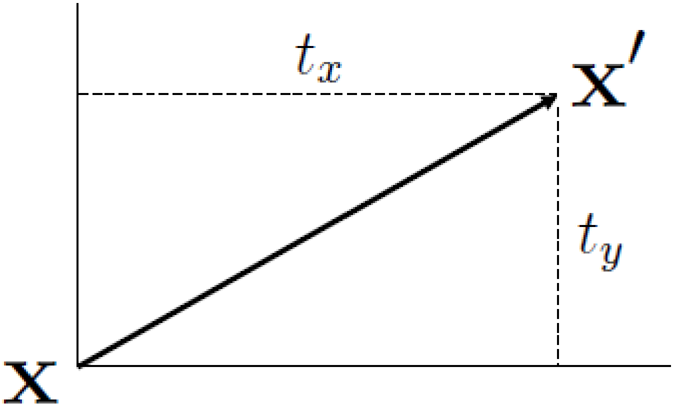
- Operations on vectors (or points)
  - Translation
  - Linear transformation
    - Scale
    - Shear
    - Rotation
    - Any combination of these
  - Affine transformation
    - Linear transformation followed by translation

2D translation

- Translation of vector **x** to **x'** under translation **t**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

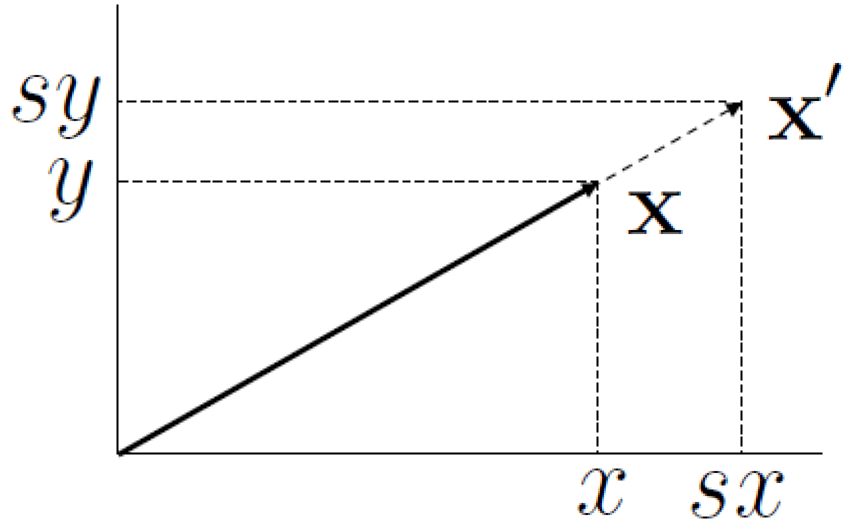


2D uniform scale

- Scale *x* and *y* the same

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = s\mathbf{I}\mathbf{x}$$

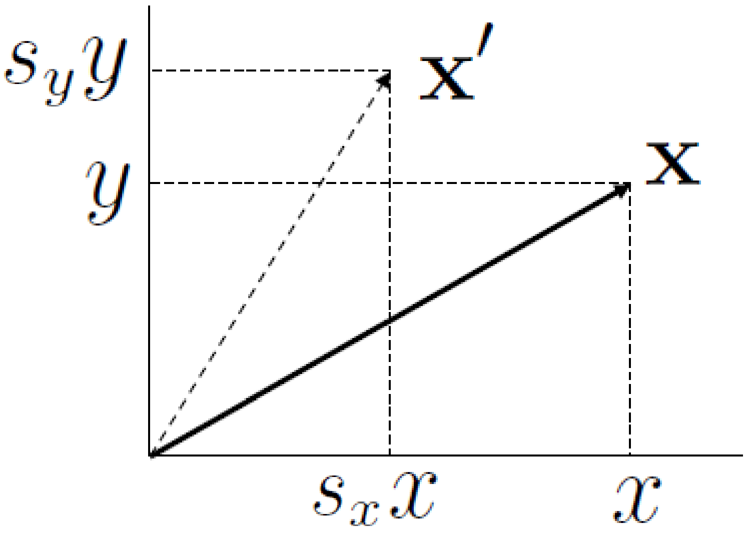


2D nonuniform scale

- Scale *x* and *y* independently

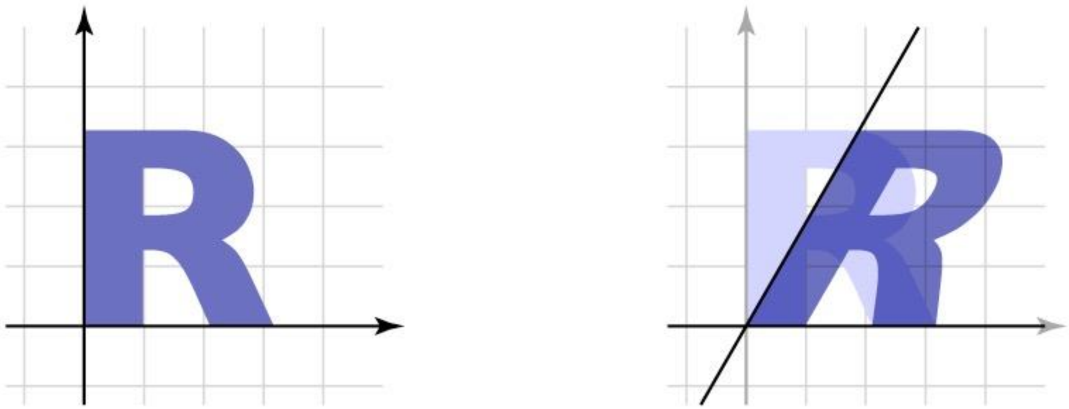
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathrm{diag}(s_x, s_y)\mathbf{x}$$



2D shear

- Shear in *x* direction (horizontal)



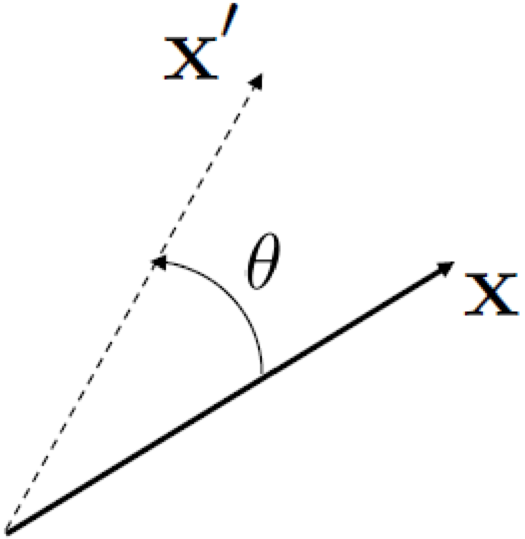
$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

2D rotation

- Positive angles rotate counterclockwise

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathrm{R}(\theta)\mathbf{x}$$



Inverse 2D rotation

- Positive angles rotate counterclockwise

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathrm{R}(\theta)\mathbf{x}$$

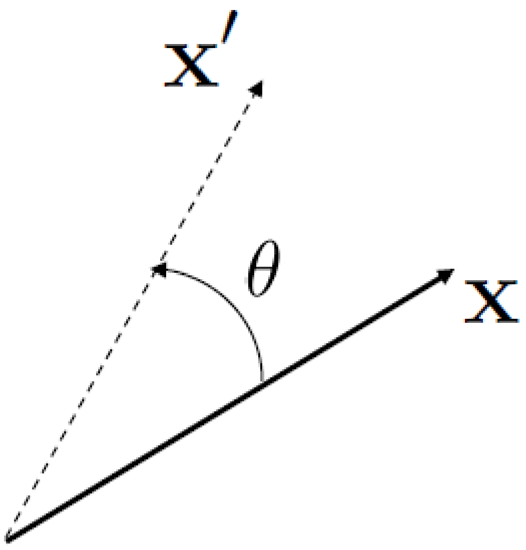
- Negative angles rotate clockwise

$$\mathbf{x} = \mathrm{R}(-\theta)\mathbf{x}'$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

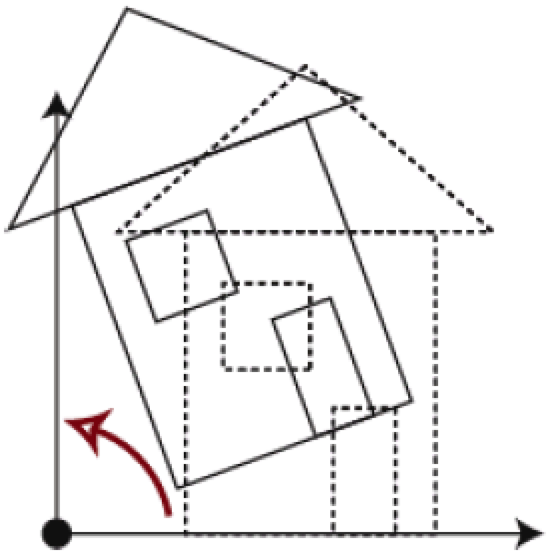
$$\mathbf{x} = \mathrm{R}(\theta)^\top \mathbf{x}'$$



$$\mathrm{R}(-\theta) = \mathrm{R}(\theta)^\top$$

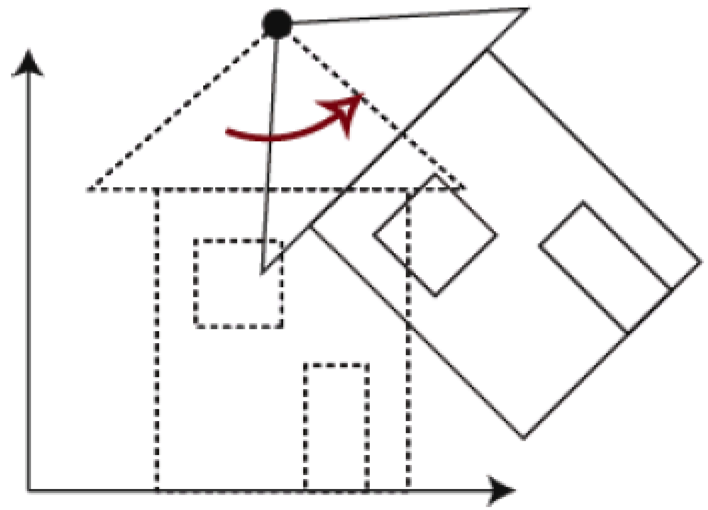
$$\mathrm{R}(-\theta)^\top = \mathrm{R}(\theta)$$

2D rotation about a point



Rotation around origin:

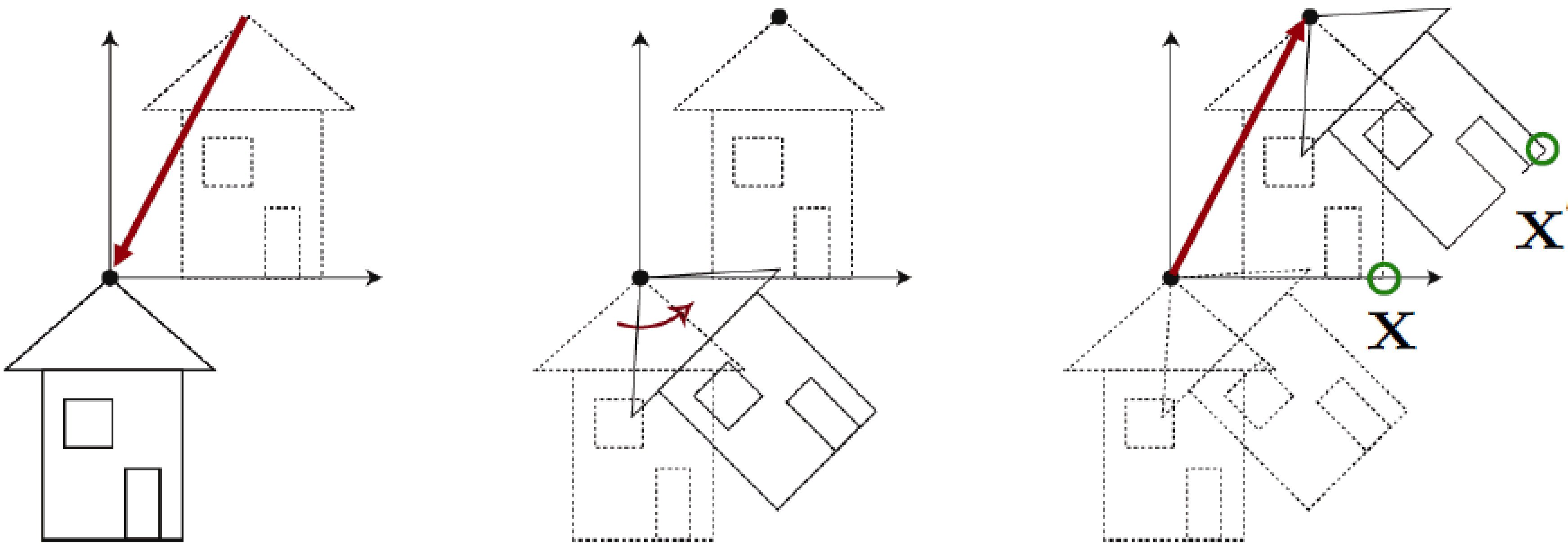
$$x' = Rx$$



Rotation around pivot point:

$$x' = ?$$

2D rotation about a point



1. Translate point to the origin

2. Rotate about the origin

3. Translate origin back to point

2D rotation about a point

- This can be accomplished with one transformation matrix, if we use homogeneous coordinates
- A 2D point using affine homogeneous coordinates is a 3-vector with 1 as the last element

$$\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D translation using homogeneous coordinates

- 2D translation using a 3x3 matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Inverse of 2D translation is inverse of 3x3 matrix

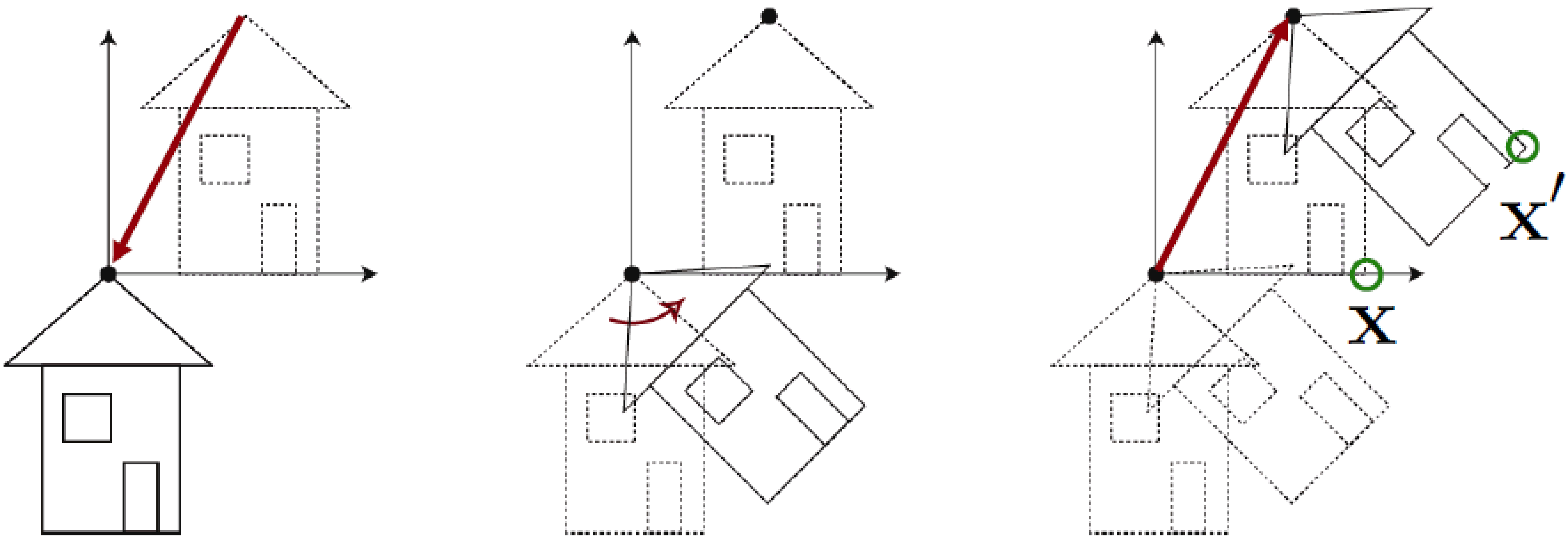
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

2D rotation using homogeneous coordinates

- 2D rotation using homogenous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D rotation about a point using homogeneous coordinates



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Important: transformation matrices are applied right to left

2D rotation about a point using homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where $\mathbf{M} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$