

## Problem Set 1

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Due on: never

**Problem 1 (10 points)**

Let  $u_1$  and  $u_2$  be vectors such that  $\|u_1\| = \|u_2\| = 1$ , and  $\langle u_1, u_2 \rangle = 0$ . For any vector  $x$ , we define  $P(x)$  as the vector  $P(x) = \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2$ .

1. How would you geometrically interpret  $P(x)$ ? (Hint: Think about projections)
2. Show that:  $\|P(x)\|^2 = \langle x, u_1 \rangle^2 + \langle x, u_2 \rangle^2$ .
3. Using parts (1) and (2), show that  $\|P(x)\| \leq \|x\|$ . When is  $\|P(x)\| = \|x\|$ ?

**Problem 2 (10 points)**

Given two column vectors  $x$  and  $y$  in  $d$ -dimensional space, the outer product of  $x$  and  $y$  is defined to be the  $d \times d$  matrix  $x \circ y = xy^\top$ .

1. Show that for any  $x$  and  $y$ ,  $x^\top(x \circ y)y = \|x\|^2\|y\|^2$ . When is this equal to  $x^\top \langle x, y \rangle y$ ?
2. Show that for any non-zero  $x$  and  $y$ , the outer product  $x \circ y$  always has rank 1.
3. Let  $x_1, \dots, x_n$  be  $n$   $d \times 1$  data vectors, and let  $X$  be the  $n \times d$  data matrix whose  $i$ -th row is the row vector  $x_i^\top$ . Show that:

$$X^\top X = \sum_{i=1}^n x_i \circ x_i$$

**Problem 3 (10 points)**

Suppose  $A$  and  $B$  are  $d \times d$  matrices which are symmetric (in the sense that  $A_{ij} = A_{ji}$  and  $B_{ij} = B_{ji}$  for all  $i$  and  $j$ ) and positive semi-definite. Also suppose that  $u$  is a  $d \times 1$  vector such that  $\|u\| = 1$ . Which of the following matrices are always positive semi-definite, no matter what  $A$ ,  $B$  and  $u$  are? Justify your answer.

1.  $10A$ .
2.  $A + B$ .
3.  $uu^\top$ .
4.  $A - B$ .
5.  $I - uu^\top$  (Hint: Write down  $x^\top(I - uu^\top)x$  in terms of some dot-products, and try using Cauchy-Schwartz.)

#### Problem 4 (10 points)

In class, we discussed how to define a *norm* or a *length* for a vector. It turns out that one can also define a norm or a length for a matrix. Two popular matrix norms are the Frobenius norm and the spectral norm. The Frobenius norm of a  $m \times n$  matrix  $A$ , denoted by  $\|A\|_F$  is defined as:

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n A_{ij}^2}$$

The spectral norm of a  $m \times n$  matrix  $A$ , denoted by  $\|A\|$  is defined as:

$$\|A\| = \max_{x \neq \mathbf{0}} \frac{\|Ax\|}{\|x\|}$$

where  $x$  is a  $n \times 1$  vector.

1. Let  $I$  be the  $n \times n$  identity matrix. What is its Frobenius norm? What is its spectral norm? Justify your answer.
2. Suppose  $A = uv^\top$  where  $u$  is a  $m \times 1$  vector and  $v$  is a  $n \times 1$  vector. Write down the Frobenius norm of  $A$  as function of  $\|u\|$  and  $\|v\|$ . Justify your answer.
3. Write down the spectral norm of  $A$  in terms of  $\|u\|$  and  $\|v\|$ . Justify your answer.