CSE 291: Communication Complexity
Homework 1, due January 28, 2019

Question 1 (Fooling sets). Let $f : X \times Y \rightarrow Z$. Recall the definition of a fooling set: a set $S \subset X \times Y$ is a fooling set if there exists $z \in Z$ such that

- $f(x, y) = z$ for all $(x, y) \in S$.
- If $(x_1, y_1), (x_2, y_2) \in S$ then either $f(x_1, y_2) \neq z$ or $f(x_2, y_1) \neq z$.

(a) Prove that if $S$ is a fooling set for $f$ then $D(f) \geq \log |S|$.

(b) Assume that $f$ is a partial function. Extend the definition of a fooling set to partial functions, and prove the same bound.

Question 2 (Inner product). The inner product function $IP : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$ is defined as $IP(x, y) = \langle x, y \rangle \mod 2$. Let $M = M_{IP}$ be its corresponding $2^n \times 2^n$ matrix.

(a) Prove that the rank of $M$ over $\mathbb{F}_2$ equals $n$.

(b) Prove that the rank of $M$ over the reals is $\Omega(2^n)$ (hint: consider the function $g(x) = (-1)^{f(x)}$, and compute $(M_g)^2$).

(c) Conclude that the inner product function requires $\Omega(n)$ deterministic communication complexity.

(d) What is the size of the largest monochromatic rectangle in $IP$? use this to give an alternative proof to the deterministic lower bound.
Question 3 (Partition vs communication). Let \( f : X \times Y \to Z \). We are interested in the relation between two measures:

- The deterministic communication complexity of \( f \), namely \( D(f) \).
- The partition number of \( f \), denoted \( P(f) \), which is the minimal number \( N \) such that \( M_f \) can be partitioned into \( N \) monochromatic rectangles.

All logarithms below are in base two. We will prove that

\[
\log P(f) \leq D(f) \leq O(\log^2 P(f)).
\]

(a) Prove that \( \log P(f) \leq D(f) \).

Assume \( P(f) = N \) and consider a partition of \( M_f \) into monochromatic rectangles \( R_1, \ldots, R_N \) where \( R_i = A_i \times B_i \). Define a graph \( G = (V, E) \) as follows. The vertices are \( V = \{1, \ldots, N\} \). There is an edge between nodes \( i, j \) if the rectangles \( R_i, R_j \) have a row in common. That is, \( (i, j) \in E \) if \( A_i \cap A_j \neq \emptyset \).

(b) Given a row \( x \) of \( M_f \), let \( C_x = \{i \in [N] : x \in A_i\} \). Prove that \( C_x \) is a clique in \( G \).

(c) Given a column \( y \) of \( M_f \), let \( I_y = \{i \in [N] : y \in B_i\} \). Prove that \( I_y \) is an independent set in \( G \).

(d) Use the Clique-vs-Independent-Set protocol of Yannakakis to deduce that \( D(f) \leq O(\log^2 P(f)) \).

Question 4 (Monotone Karchmer-Wigderson games). Let \( f : \{0,1\}^n \to \{0,1\} \) be a monotone boolean function. That is, for any \( x, y \in \{0,1\}^n \), if \( x_i \leq y_i \) for all \( i \in [n] \) then \( f(x) \leq f(y) \).

Any monotone function can be computed by a monotone formula with only AND and OR gates (that is, no negations are allowed). Denote by mon-depth(\( f \)) the minimum depth of a monotone formula computing \( f \).

The monotone Karchmer-Wigderson game for \( f \), denoted mon-KW_\( f \), is the following game: Alice gets an input \( x \) where \( f(x) = 0 \), Bob gets an input \( y \) where \( f(y) = 1 \). Their goal is to find \( i \in [n] \) such that \( x_i = 0, y_i = 1 \).

(a) Prove that such \( i \) always exists.

(b) Prove that \( \text{mon-depth}(f) = D(\text{mon-KW}_f) \).