**CSE 203B W19 Homework 1 Solution**

**Due Time : 12:00pm, Th Jan. 24, 2019 Submit to Gradescope**
**Gradescope:** `https://gradescope.com/`

In this homework assignment, we work on convex set examples including Voronoi diagram (2.7, 2.9), quadratic function (2.10), general sets (2.12), probability (2.15) and dual cones (2.30). The first assignment reviews the concept of the null space in linear algebra. The second assignment works on the set enumeration. The last two assignements utilize Matlab to practice the set operations. The six exercises worth 1 point each (graded by completion). The first two assignment worth 3 points each (graded by content). The last two assignment worth 4 points each (graded by content). Thus, the total points are 20.

**I Exercises from textbook chapter 2:** 2.7, 2.9, 2.10, 2.12, 2.15, 2.30

**II Assignments**

II. 1. Range and Null space: Given an expression $Ax$ where $A = \begin{pmatrix} 2 & 4 & 6 & 8 & 1 \\ 1 & -1 & 2 & 0 & -1 \\ 3 & 0 & 7 & 4 & 0 \end{pmatrix}$. Identify the null space of $A$ and $A^T$. Express your solution with the basis of the space.

Apply Gaussian elimination on rows of $A$ to get the reduced row echelon form (rref). First divide row 1 by 2, then subtract row 1 from row 2 and subtract 3 times of row 1 from row 3.

$$A \to A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 1/2 \\ 0 & -3 & -1 & -4 & -3/2 \\ 0 & -6 & -2 & -8 & -3/2 \end{pmatrix}$$

Divide row 2 by -3 then add 6 times row 2 to row 3.

$$A_1 \to A_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 1/2 \\ 0 & 1 & 1/3 & 4/3 & 1/2 \\ 0 & 0 & 0 & 0 & 3/2 \end{pmatrix}$$

Divide row 3 by 3/2 then subtract 1/2 row 3 from row 1 and row 2 to set all the other elements in col 5 to zeros.

$$A_2 \to A_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1/3 & 4/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Similarly subtract 2 times of row 2 from row 1 then we have the rref of $A$.

$$A_3 \to A_4 = \begin{pmatrix} 1 & 0 & 7/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & 4/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We could tell that col 1, 2 and 5 are independent, which span the range of $A$. Rewrite $A_4 x = 0$ as follows

$$\begin{aligned} x_1 &= -7/3x_3 - 4/3x_4 \\ x_2 &= -1/3x_3 - 4/3x_4 \\ x_5 &= 0 \end{aligned}$$

Knowing that $x_3$ and $x_4$ are free variables, so that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7/3x_3 - 4/3x_4 \\ -1/3x_3 - 4/3x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -7/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4/3 \\ -4/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The null space of $A$ is defined as

$$\mathcal{N}(A) = span\Big\{ \begin{bmatrix} -7/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4/3 \\ -4/3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Big\}$$

The rank of $A$ is 3, so the columns of $A^T$ are independent and we could get its rref as

$$A^T \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The null space of $A^T$ is

$$\mathcal{N}(A^T) = span\Big\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Big\}$$

II. 2. Implicit and explicit description of the convex set. Given set $\{x | Ax \leq b\}$ where the equation $Ax \leq b$ is described as follows.

$$\begin{aligned} -x_1 + 5x_2 &\leq 8 \\ 4x_1 - x_2 &\leq 6 \\ -x_2 &\leq 2 \\ -6x_1 - 5x_2 &\leq 13 \end{aligned}$$

Express the set with explicit enumeration of a basis.

Find the intersection points $u_1 = (2, 2)$, $u_2 = (-3, 1)$, $u_3 = (-1/2, -2)$, $u_4 = (1, -2)$ as shown in Figure 1.

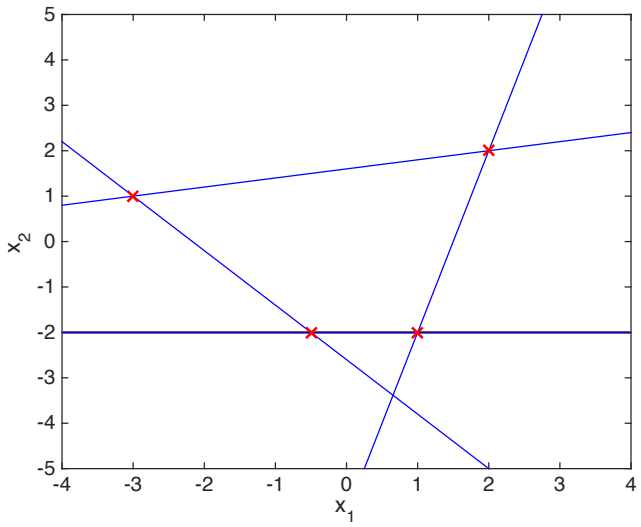


Figure 1: The convex set and intesection points.

The explicit expression of the set is

$$S = \{\theta_1 u_1 + \theta_2 u_2 + \theta_3 u_3 + \theta_4 u_4 | \Sigma_{i=1}^4 \theta_i = 1 \text{ and } \theta_i \geq 0 \text{ for } i = 1, \ldots, 4\}$$

II. 3. Use Matlab or other mathematical tools to implement. Find the range and nullspace of matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 2 & 3 & 5 \end{pmatrix}$ and its transpose. Validate the results with your own calculation.

Call $eig()$ in Matlab and we could get the eigenvalue decomposition of $A$,

$$\Lambda = \begin{pmatrix} 8.1231 & & \\ & 0.0000 & \\ & & -0.1231 \end{pmatrix}, Q = \begin{pmatrix} -0.2941 & -0.5774 & -0.3667 \\ -0.5882 & -0.5774 & -0.7333 \\ -0.7533 & 0.5774 & 0.5726 \end{pmatrix} \quad (1)$$

where $A = Q\Lambda Q^{-1}$.

So the range of $A$ could be expressed with the span of eigenvectors corresponding to nonzero eigenvalues. Any vector in $Q$ which satisfies $Av = 0$ is in the nullspce.

$$\mathcal{R}(A) = \{ \begin{bmatrix} -0.2941 \\ -0.5882 \\ -0.7533 \end{bmatrix}, \begin{bmatrix} -0.3667 \\ -0.7333 \\ 0.5726 \end{bmatrix} \}, \mathcal{N}(A) = \{ \begin{bmatrix} -0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix} \}$$

The range and nullspace of $A^T$ can be obtained in the same way using $eig()$

$$\Lambda = \begin{pmatrix} 8.1231 & & \\ & 0.0000 & \\ & & -0.1231 \end{pmatrix}, Q = \begin{pmatrix} -0.3571 & -0.8944 & 0.7767 \\ -0.4574 & 0.4472 & -0.6064 \\ -0.8144 & 0.0000 & 0.1703 \end{pmatrix} \quad (2)$$

where $A^T = Q\Lambda Q^{-1}$.

$$\mathcal{R}(A^T) = \{ \begin{bmatrix} -0.3571 \\ -0.4574 \\ -0.8144 \end{bmatrix}, \begin{bmatrix} 0.7767 \\ -0.6064 \\ 0.1703 \end{bmatrix} \}, \mathcal{N}(A^T) = \{ \begin{bmatrix} -0.8944 \\ 0.4472 \\ 0.0000 \end{bmatrix} \}$$

II. 4. Use Matlab or other mathematical tools to implement. Find the range and nullspace of matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$ and its transpose. Validate the results with your own calculation.

We apply SVD function on matrix $A$,

$$\begin{aligned} U &= \begin{pmatrix} 0.5774 & -0.3853 & 0.7199 & 0.0000 \\ 0.0000 & -0.7199 & -0.3853 & -0.5774 \\ 0.5774 & 0.5526 & -0.1673 & -0.5774 \\ 0.5774 & -0.1673 & -0.5526 & 0.5774 \end{pmatrix} \in \mathbb{R}^{4\times 4} \\ \Sigma &= \begin{pmatrix} 2.6458 & 0 & 0 & 0 & 0 \\ 0 & 2.3028 & 0 & 0 & 0 \\ 0 & 0 & 1.3028 & 0 & 0 \\ 0 & 0 & 0 & 0.0000 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 5} \\ V &= \begin{pmatrix} -0.4364 & -0.5526 & -0.1673 & -0.6769 & -0.1342 \\ 0.0000 & -0.7199 & 0.3853 & 0.4123 & 0.4041 \\ 0.6547 & 0.0000 & 0.0000 & 0.5291 & -0.5398 \\ 0.4364 & 0.3853 & 0.7199 & -0.2646 & 0.2699 \\ 0.4364 & 0.1673 & -0.5526 & 0.1477 & 0.6741 \end{pmatrix} \in \mathbb{R}^{5\times 5} \end{aligned}$$

where $A = U\Sigma V^*$, $UU^* = I$ and $VV^* = I$.

The right-singular vectors in $V$ corresponding to zero singular values of $A$ span the nullspace of $A$, the left-singular vectors in $U$ corresponding to the nonzero singular values span the range of $A$. So we have

$$\mathcal{R}(A) = \{ \begin{bmatrix} 0.5774 \\ 0.0000 \\ 0.5774 \\ 0.5774 \end{bmatrix}, \begin{bmatrix} -0.3853 \\ -0.7199 \\ 0.5526 \\ -0.1673 \end{bmatrix}, \begin{bmatrix} 0.7199 \\ -0.3853 \\ -0.1673 \\ -0.5526 \end{bmatrix} \}, \mathcal{N}(A) = \{ \begin{bmatrix} -0.6769 \\ 0.4123 \\ 0.5291 \\ -0.2646 \\ 0.1477 \end{bmatrix}, \begin{bmatrix} -0.1342 \\ 0.4041 \\ -0.5398 \\ 0.2699 \\ 0.6741 \end{bmatrix} \}$$

For $A$ transpose, we have $A^T = (U\Sigma V^*)^T = V\Sigma^T U^*$ and similarly

$$\mathcal{R}(A^T) = \{ \begin{bmatrix} -0.4364 \\ 0.0000 \\ 0.6547 \\ 0.4364 \\ 0.4364 \end{bmatrix}, \begin{bmatrix} -0.5526 \\ -0.7199 \\ 0.0000 \\ 0.3853 \\ 0.1673 \end{bmatrix}, \begin{bmatrix} -0.1673 \\ 0.3853 \\ 0.0000 \\ 0.7199 \\ -0.5526 \end{bmatrix} \}, \mathcal{N}(A^T) = \{ \begin{bmatrix} 0.0000 \\ -0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix} \}$$