CSE 203B: Convex Optimization
Week 7 Discuss Session

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Unconstrained Minimization Problems

- Assume that the objective function is ***strongly convex*** on S, there exists an $m > 0$ such that

$$\nabla^2 f(x) \succcurlyeq mI \quad (1)$$

for all $x \in S$.

- Upper bound on $\nabla^2 f(x)$, there exist a constant M such that

$$\nabla^2 f(x) \preccurlyeq MI \quad (2)$$

for all $x \in S$.

In class we are given two inequality condition for the optimal

$$\frac{1}{2M}\|\nabla f(x)\|_2 \leq \|x - x^*\|_2 \leq \frac{2}{m}\|\nabla f(x)\|_2$$

$$\frac{1}{2M}\|\nabla f(x)\|^2{}_2 \leq f(x) - p^* \leq \frac{1}{2m}\|\nabla f(x)\|^2{}_2$$

How to prove?

Unconstrained Minimization Problems

- Taylor expansion with any $x, y \in S$.

$$f(y) = f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T\nabla^2 f(z)(y-x)$$

for z on the line segment between x and y.

  - Consider the strong convexity $\nabla^2 f(x) \succcurlyeq mI$ for any $x \in S$

$$f(y) \geq f(x) + \nabla f(x)^T(y-x) + \frac{m}{2}||y-x||_2^2$$

$$= f(x) + \frac{m}{2}\left|\left|y - x + \frac{1}{m}\nabla f(x)\right|\right|_2^2 - \frac{1}{2m}||\nabla f(x)||_2^2$$

$$\geq f(x) - \frac{1}{2m}||\nabla f(x)||_2^2$$

  - It holds for any $y \in S$, let $y = x^*$ be the optimal

$$p^* = f(x^*) \geq f(x) - \frac{1}{2m}||\nabla f(x)||_2^2$$

$$\Rightarrow f(x) - p^* \leq \frac{1}{2m}||\nabla f(x)||_2^2$$

Unconstrained Minimization Problems

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$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)$$

for z on the line segment between x and y.

  - Consider the upper bound $\nabla^2 f(x) \preccurlyeq MI$ for any $x \in S$

$$f(y) \le f(x) + \nabla f(x)^T (y - x) + \frac{M}{2} ||y - x||_2^2$$

try to minimize the rhs, the minimum is achieved at $y = x - \frac{1}{M} \nabla f(x)$

$$f\left(x - \frac{1}{M} \nabla f(x)\right) \le f(x) - \frac{1}{2M} ||\nabla f(x)||_2^2$$

  - Let x^* be the optimal

$$p^* = f(x^*) \le f\left(x - \frac{1}{M} \nabla f(x)\right) \le f(x) - \frac{1}{2M} ||\nabla f(x)||_2^2$$

$$\Rightarrow \frac{1}{2M} ||\nabla f(x)||_2^2 \le f(x) - p^*$$

$$\frac{1}{2M} \|\nabla f(x)\|^2{}_2 \le f(x) - p^* \le \frac{1}{2m} \|\nabla f(x)\|^2{}_2$$

Unconstrained Minimization Problems

- Taylor expansion with any $x, y \in S$.

$$f(y) = f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)$$

for z on the line segment between x and y.

  - Consider the strong convexity $\nabla^2 f(x) \succcurlyeq mI$ for any $x \in S$, we have

$$p^* = f(x^*) \geq f(x) + \nabla f(x)^T(x^* - x) + \frac{m}{2}\left|\left|x^* - x\right|\right|_2^2$$

$$\geq f(x) - \left|\left|\nabla f(x)\right|\right|_2 \left|\left|x^* - x\right|\right|_2 + \frac{m}{2}\left|\left|x^* - x\right|\right|_2^2 \quad \textit{Cauchy-Schwarz inequality}$$

  - Since $p^* \leq f(x)$ for all $x \in S$

$$-\left|\left|\nabla f(x)\right|\right|_2 \left|\left|x^* - x\right|\right|_2 + \frac{m}{2}\left|\left|x^* - x\right|\right|_2^2 \leq 0$$

$$\Rightarrow \left|\left|x^* - x\right|\right|_2 \leq \frac{2}{m}\left|\left|\nabla f(x)\right|\right|_2$$

Unconstrained Minimization Problems

- Taylor expansion with any $x, y \in S$.

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)$$

for z on the line segment between x and y.

  - Consider the first-order condition for any $x \in S$ and x^* is optimal

$$p^* = f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x)$$
$$\geq f(x) - \left|\left|\nabla f(x)\right|\right|_2 \left|\left|x^* - x\right|\right|_2 \quad \textit{Cauchy-Schwarz inequality}$$

  - We already prove that

$$p^* = f(x^*) \leq f(x) - \frac{1}{2M}\left|\left|\nabla f(x)\right|\right|_2^2$$
$$\Rightarrow -\left|\left|\nabla f(x)\right|\right|_2 \left|\left|x^* - x\right|\right|_2 \leq -\frac{1}{2M}\left|\left|\nabla f(x)\right|\right|_2^2$$
$$\Rightarrow \frac{1}{2M}\left|\left|\nabla f(x)\right|\right|_2 \leq \left|\left|x^* - x\right|\right|_2$$

$$\frac{1}{2M}\|\nabla f(x)\|_2 \leq \|x - x^*\|_2 \leq \frac{2}{m}\|\nabla f(x)\|_2$$