Linear Algebra Review

Xinyuan Wang

01/09/2019

Contents

Matrix and Vector

- A vector $x \in \mathbb{R}^n$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- A matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

- How to solve a linear system $Ax = b \Longrightarrow \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$?

Linear Systems

- Solve $Ax = b$, equivalent to find the n unknown variables $x_1, \dots, x_n$ satisfying equations

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{gathered}
$$

- The LHS defines a linear combination of A's column vectors

$$x_1 c_1(A) + x_2 c_2(A) + \cdots + x_n c_n(A) = b$$

  where $c_i(A)$ is the i^{th} column of A.

- When does the system have a solution?

Linear Systems

- A necessary and sufficient condition that $Ax = b$ has a solution is that $b \in S(c_1(A), \dots, c_n(A))$, which is a space in $\mathbb{R}^m$ spanned by the columns of A.
- Gaussian elimination with row operations
  - The solution won't be changed

$$[a_{11} \quad \cdots \quad a_{1n}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b_1$$

$$\vdots$$

$$[a_{m1} \quad \cdots \quad a_{mn}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b_m$$

Linear Systems

- Excise: apply Gaussian elimination on matrix

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

and find the solution of $Ax = b$ where

1) $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ 2) $b = \begin{bmatrix} 1 \\ -2 \\ 1.5 \end{bmatrix}$ does solution exist?

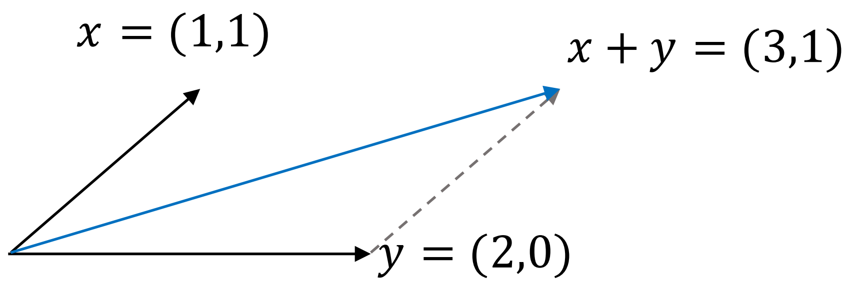
$$A \Longrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ \boxed{0 \quad 0 \quad 0} \end{pmatrix}$$

Case 1) b is not in the space spanned by columns in A

Case 2) the solution is not unique

Norms

- Given a vector $x \in \mathbb{R}^n$, its norm is given by
  - Euclidean norm (L_2-norm): $\|x\|_2 = \langle x, x \rangle = (x_1^2 + \cdots + x_n^2)^{1/2}$
  - Sum-absolute-value (L_1-norm): $\|x\|_1 = |x_1| + \cdots + |x_n|$
  - Chebyshev (L_∞-norm): $\|x\|_\infty = \max_i |x_i|$ for $i = 1, \ldots, n$
- The vector norm satisfies
  - $\|cx\| = |c|\|x\|$
  - $\|x\| = 0 \Leftrightarrow x = 0$
  - $\|x + y\| \leq \|x\| + \|y\|$ (**triangle inequality**)



Matrix Subspaces

- The **range** of matrix $A \in \mathbb{R}^{m\times n}$ is $\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$ or column subspace $col(A)$
- Row subspace $row(A) = \{A^T y \mid y \in \mathbb{R}^m\}$
- The **nullspace** of A is defined to be $\mathcal{N}(A) = \{x \mid Ax = 0\}$
  - $\mathcal{N}(A) \perp row(A)$
  - $\dim(\mathcal{N}(A)) + \dim(row(A)) = n$
  - Analogy for column space

Excises: what is the range of the matrix below? Nullspace?

1) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ 2) $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$

Case 1) $\mathcal{N}(A) = 0$

Case 2) example of nullspace vector of A,

$$y = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \in \mathcal{N}(A)$$

Rank

- Definition: the **rank** of any matrix $A \in \mathbb{R}^{m \times n}$, denoted by $r(A)$, is the dimension of its row and column subspaces.
- Properties
  - $r(A) \leq \min(m, n)$
  - $r(A) = r(A^T)$
  - $r(AB) \leq \min\big(r(A), r(B)\big)$
  - $r(A + B) \leq r(A) + r(B)$
  - A has full rank if $r(A) = \min(m, n)$
  - If $r(A) < \min(m, n)$, then the rows/columns of A are not linearly independent

Matrix Inverse

- A square matrix $A \in \mathbb{R}^{n\times n}$ is **invertible** if $r(A) = n$.
  - There exists $B \in \mathbb{R}^{n\times n}$ so that $AB = I$ and $B = A^{-1}$ is unique
  - A if full rank and $\dim\big(\mathcal{N}(A)\big) = 0$.
- Solve a linear system $Ax = b$
  - Has a solution for any $b \in \mathbb{R}^n$ (consistent)
  - The solution $x = A^{-1}b$ is unique
  - If $b = 0$, then $x = 0$

Excises: does the linear system have unique solution?

a) $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Yes, $x = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

b) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ No, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathcal{N}\left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)$

any nonzero solution

Determinants

- Let $A \in \mathbb{R}^{n\times n}$, define the **determinant** of A to be the value

$$\det A = \sum_{\sigma} \left(\prod_{i=1}^{n} a_{i\sigma(i)}\right) \cdot \operatorname{sgn} \sigma$$

where σ is a permutation of integers $\{1, 2, \ldots, n\}$ and

1) Σ_σ denotes the sum over all permutations
2) **Transposition**: exchange of only two elements of an ordered list
3) $\operatorname{sgn} \sigma =$ sign of $\sigma =$
$$\begin{cases} +1, & \text{the number of transpositions to bring } \sigma \text{ to } \{1,2,\ldots,n\} \text{ is even} \\ -1, & \text{the number of transpositions to bring } \sigma \text{ to } \{1,2,\ldots,n\} \text{ is odd} \end{cases}$$

Compute Determinants

- Let $A \in \mathbb{R}^{n\times n}$, for any row i and column j of A define the $ij - minor$

$$M_{ij} = \det A \Big|_{i^{th}\, row, j^{th}\, column\, removed}$$

1) For any fixed row k

$$\det A = \sum_{j=1}^{n} a_{kj} (-1)^{k+j} M_{kj}$$

2) For any fixed column m

$$\det A = \sum_{j=1}^{n} a_{jm} (-1)^{m+j} M_{jm}$$

Excises: compute the determinant of

$$\begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix} \qquad \mathbf{-14}$$

Determinants

- Let $A \in \mathbb{R}^{n\times n}$, properties of determinant
  - If $r(A) < n$, then $\det A = 0$
  - If A is triangular, then $\det A = \prod_{i=1}^{n} a_{ii}$
  - If $r(A) = n$, then $\det A \neq 0$
- **If $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{\boldsymbol{n\times n}}$, then $\det \boldsymbol{AB} = \det \boldsymbol{A} \det \boldsymbol{B}$.**

Excises: calculate the determinants of A, B, AB

$$A = \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$$

- Define the adjugate (adjoint) $\hat{A}_{ij} = (-1)^{i+j} M_{ji}$, if A is invertible then $A^{-1} = \frac{1}{\det A}\hat{A}$

Excises: find the adjugate and inverse of

$$\begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 1 & -4 \\ -2 & 2 \end{pmatrix}, \det A = -6$$

Eigenvalues

- Let $A \in \mathbb{R}^{n\times n}$, $Ax = \lambda x$ leads to $\det(A - \lambda I) = 0$.
- If A is **symmetric**, then it could be decomposed into

$$A = Q\Lambda Q^T = \sum_{i=1}^{n} \lambda_i \boldsymbol{q_i} \boldsymbol{q_i^T},$$

where $Q \in \mathbb{R}^{n\times n}$ is **orthogonal** which satisfies $Q^T Q = I$, and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Each vector $\boldsymbol{q_i}$ is eigenvector corresponds to eigenvalue λ_i.

- If A is not symmetric but diagonalizable, then

$$A = Q\Lambda Q^{-1}$$

where Q is not necessarily orthogonal.

- For general $A \in \mathbb{R}^{m\times n}$, singular value decomposition $A = U\Lambda V^T$

Eigenvalues

- Let $A \in \mathbb{R}^{n \times n}$, the properties of its eigenvalues

$$\det A = \prod_{i=1}^{n} \lambda_i \qquad \mathbf{tr}\, A = \sum_{i=1}^{n} \lambda_i$$

- The largest and smallest eigenvalues satisfy

$$\lambda_{max}(A) = \sup_{x \neq 0} \frac{x^T A x}{x^T x} \qquad \lambda_{min}(A) = \inf_{x \neq 0} \frac{x^T A x}{x^T x}$$

Excises: compute the eigenvalues, eigenvectors, determinant of following matrices.

a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\qquad \lambda = 1{,}3, \det A = 3$

b) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ $\qquad \lambda = 2 \pm 1i, \det A = 5$

- Read *“Convex Optimization” Appendix A by Stephen Boyd* for more relevant mathematical background.