The geometry of linear classification

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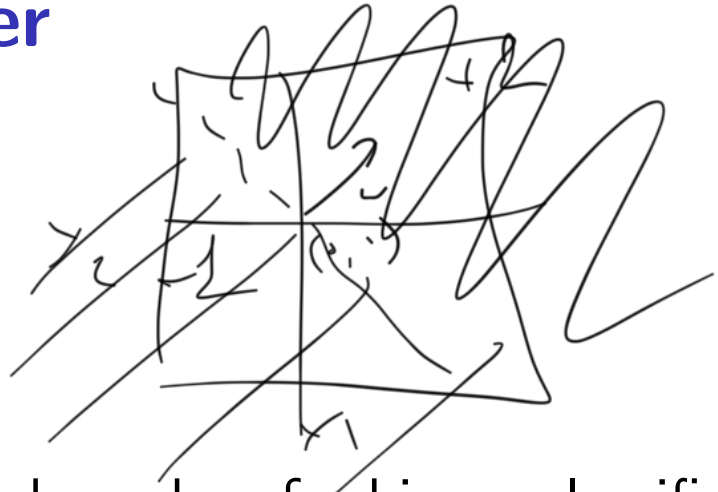
Course website:
http://cseweb.ucsd.edu/classes/wi19/cse151-b/

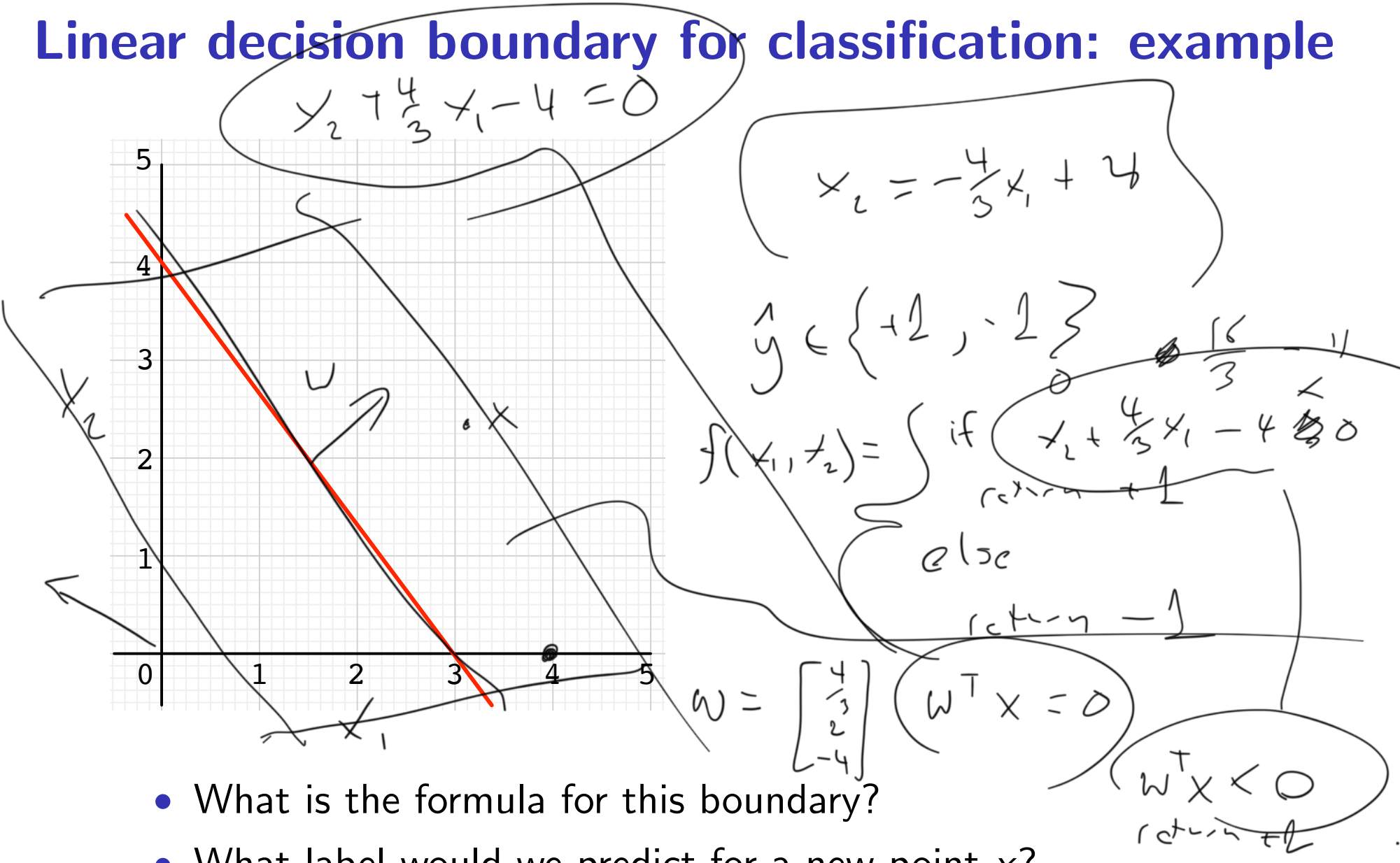
Topics we'll cover

$w^T x =$ score of +2 class



1. Linear decision boundary for binary classification
2. A loss function for classification
3. The Perceptron algorithm

Linear decision boundary for classification: example



- What is the formula for this boundary?
- What label would we predict for a new point x?

Linear classifiers

Binary classification problem: data $x \in \mathbb{R}^d$ and labels $y \in \{-1, +1\}$

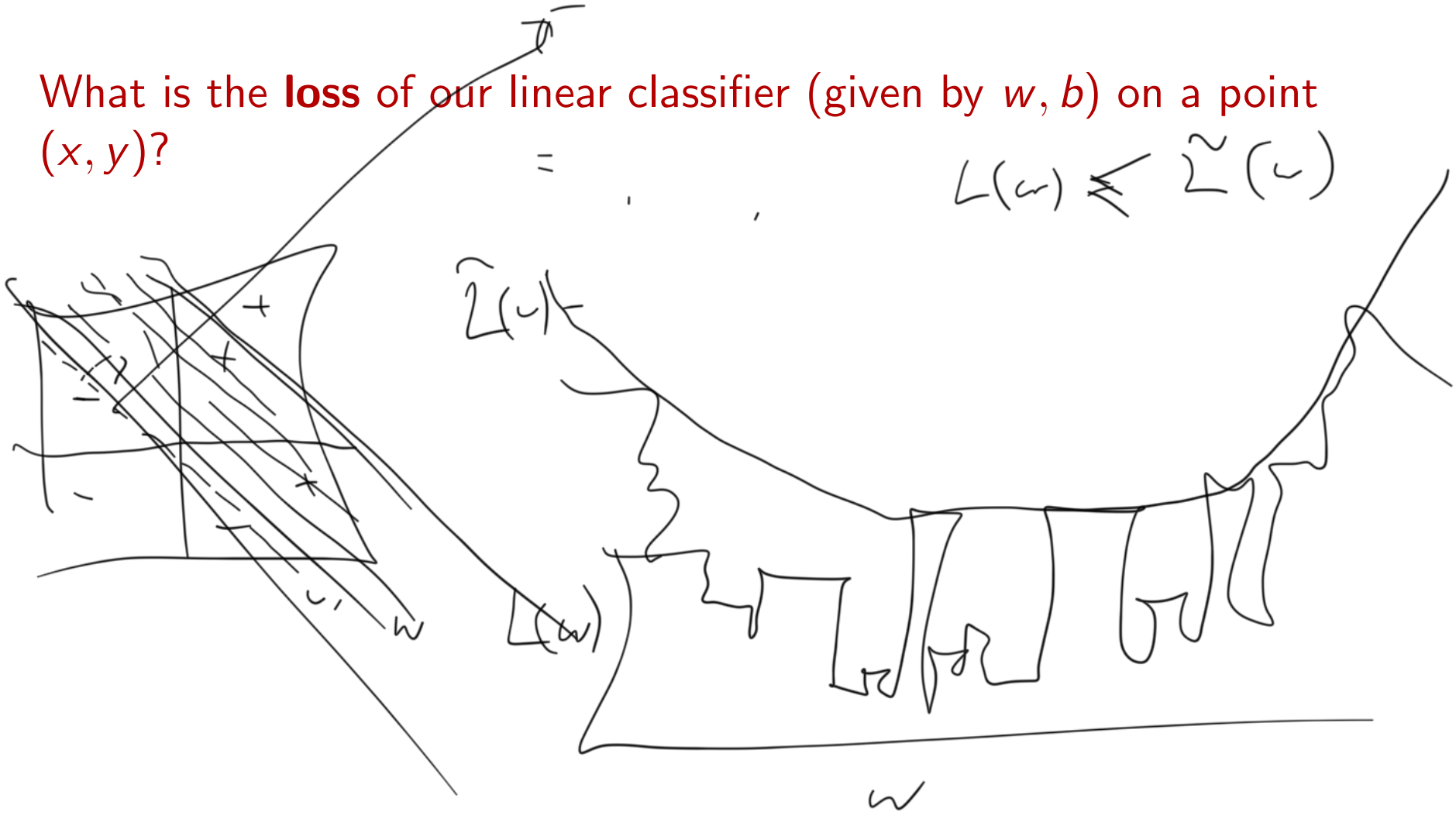
- Linear classifier:
  - Parameters: $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$
  - Decision boundary $w \cdot x + b = 0$
  - On point x, predict label $\text{sign}(w \cdot x + b)$
- If the true label on point x is y:
  - Classifier correct if $y(w \cdot x + b) > 0$

$w^T \cdot x = 0$

$\text{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{o.w.} \end{cases}$

A loss function for classification

What is the **loss** of our linear classifier (given by w, b) on a point (x, y)?

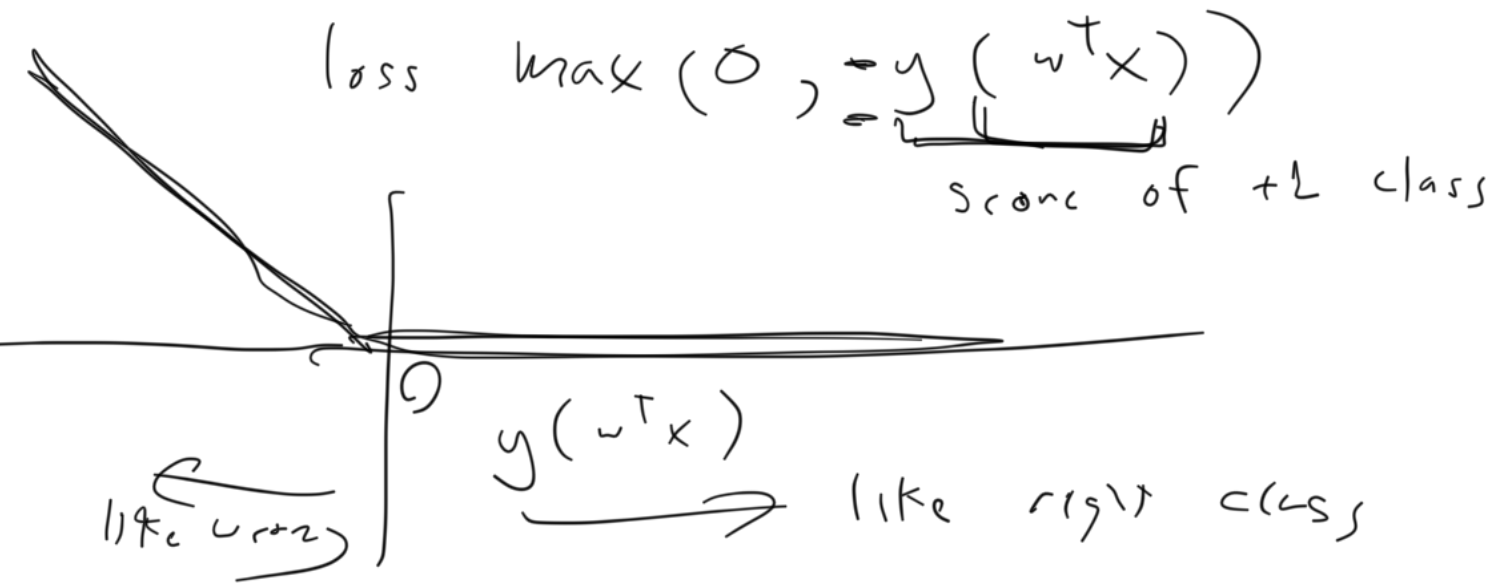


A loss function for classification

What is the **loss** of our linear classifier (given by w, b) on a point (x, y)?

One idea for a loss function:

- If $y(w \cdot x + b) > 0$: correct, no loss
- If $y(w \cdot x + b) < 0$: loss $= -y(w \cdot x + b)$



A simple learning algorithm

Fit a linear classifier w, b to the training set using **stochastic gradient descent**.

- Update w, b based on just one data point (x, y) at a time
- If $y(w \cdot x + b) > 0$: zero loss, no update
- If $y(w \cdot x + b) < 0$: loss is $-y(w \cdot x + b)$

$w = (0, 0, \ldots, 0)$
for epoch $i = 1 \ldots E$
  for $i = 2 \ldots n$
    if $y(w^T x) < 0$
      $w = w + yx$

$(w + yx)^T x = (w + x)^T x$
$= w^T x + x^T x$
$\|x\|_2^2 \geq 0$

$y = +2$
current: $w^T x < 0$
$w^T x < (w + yx)^T x$

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The Perceptron algorithm

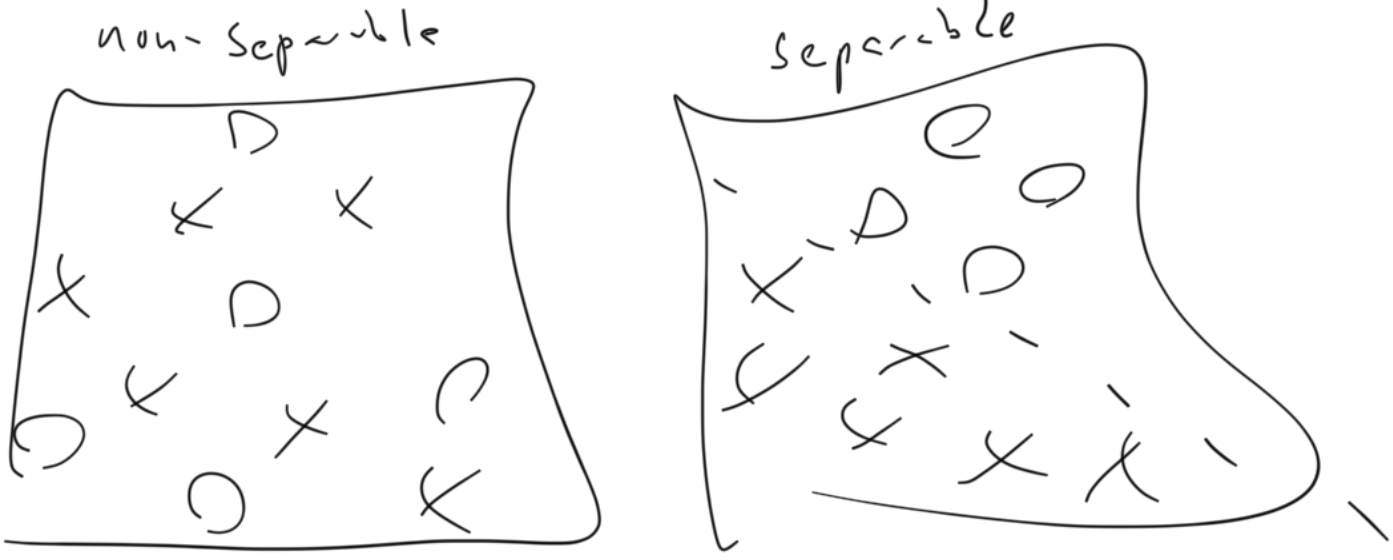
- Initialize $w = 0$ and $b = 0$
- Keep cycling through the training data (x, y):
  - If $y(w \cdot x + b) < 0$ (i.e. point misclassified):
    - $w = w + yx$
    - $b = b + y$

The learning problem: separable case

Input: training data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$

Output: linear classifier $w \in \mathbb{R}^d$ such that for every $1 \leq i \leq n$,

$$w \cdot x^{(i)} > 0 \quad \text{if } y^{(i)} = +1$$
$$w \cdot x^{(i)} < 0 \quad \text{if } y^{(i)} = -1$$



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More simply: $y^{(i)}(w \cdot x^{(i)}) > 0$ for all i.

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The **Perceptron algorithm** (Rosenblatt, 1958)

- $w = 0$
- while some (x, y) is misclassified:
  - $w = w + yx$

Perceptron: example

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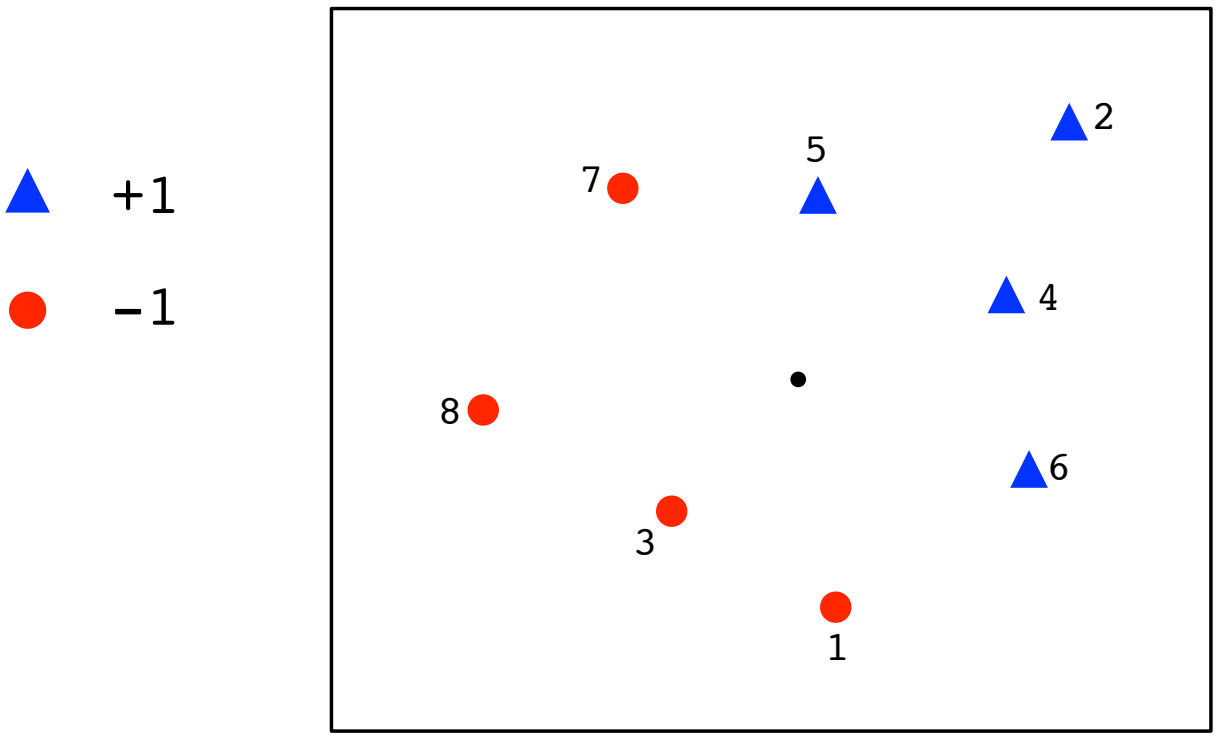

Separator: $w = 0$

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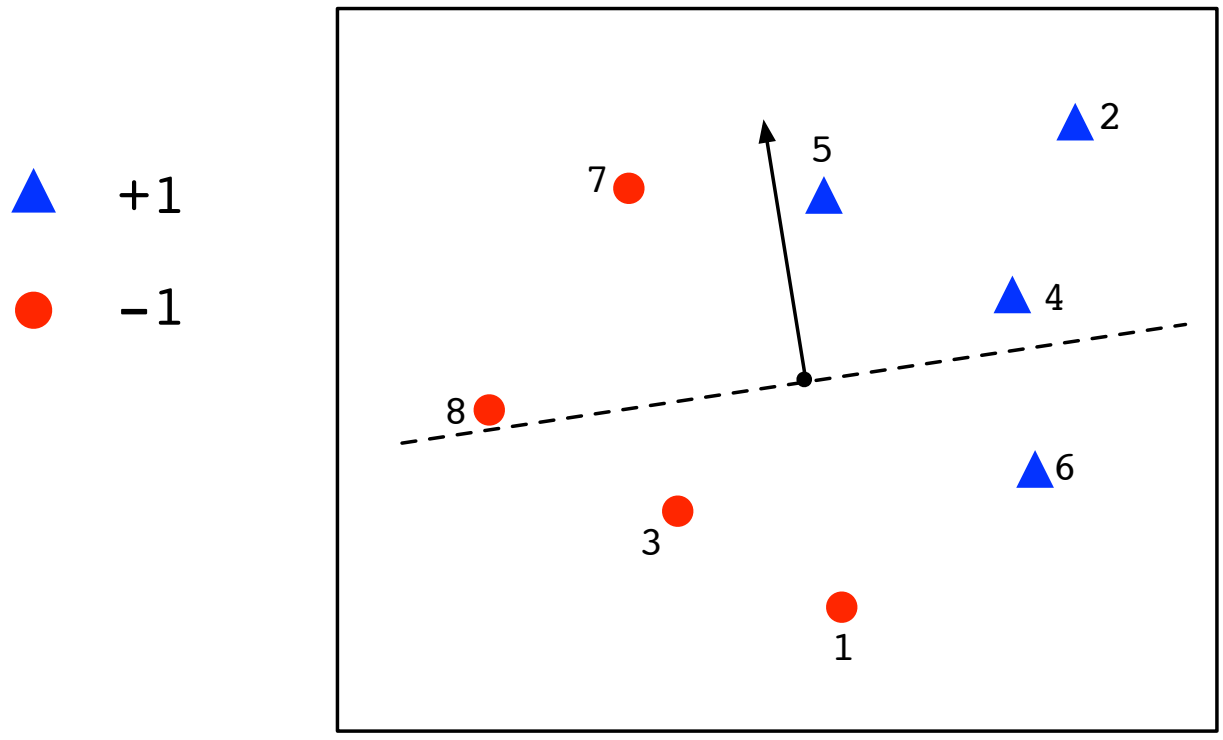

Separator: $w = -x^{(1)}$

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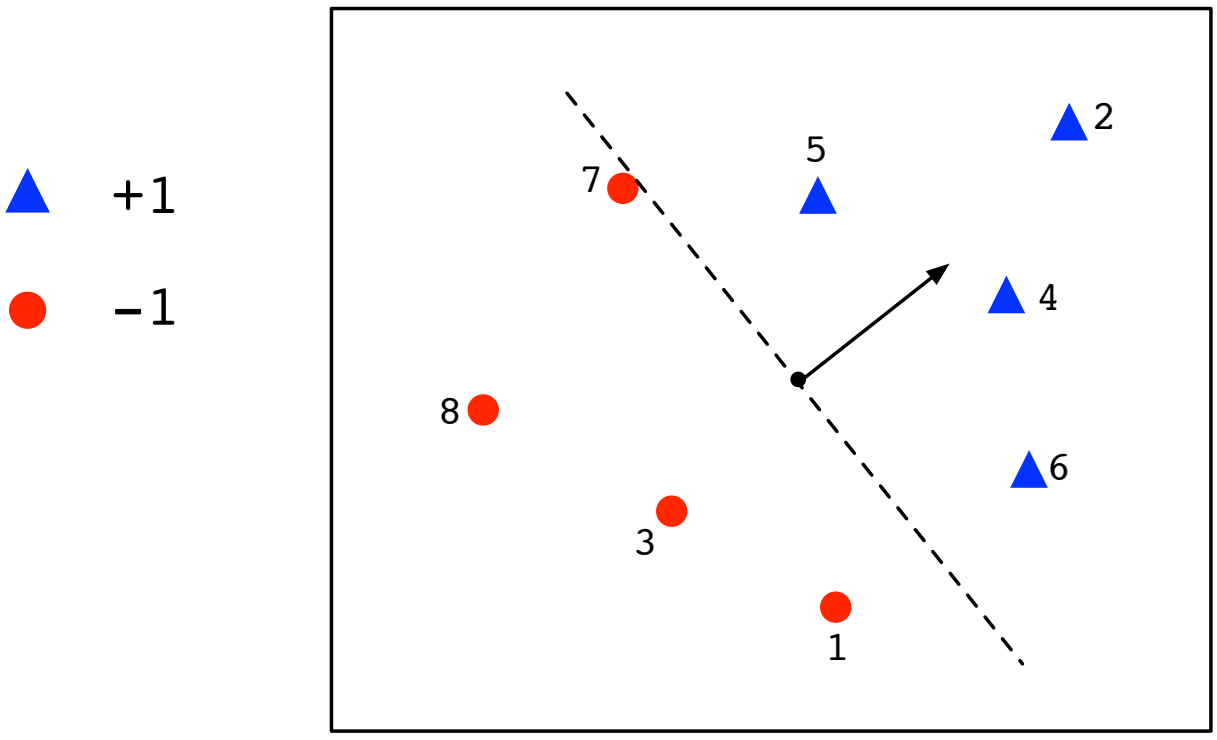
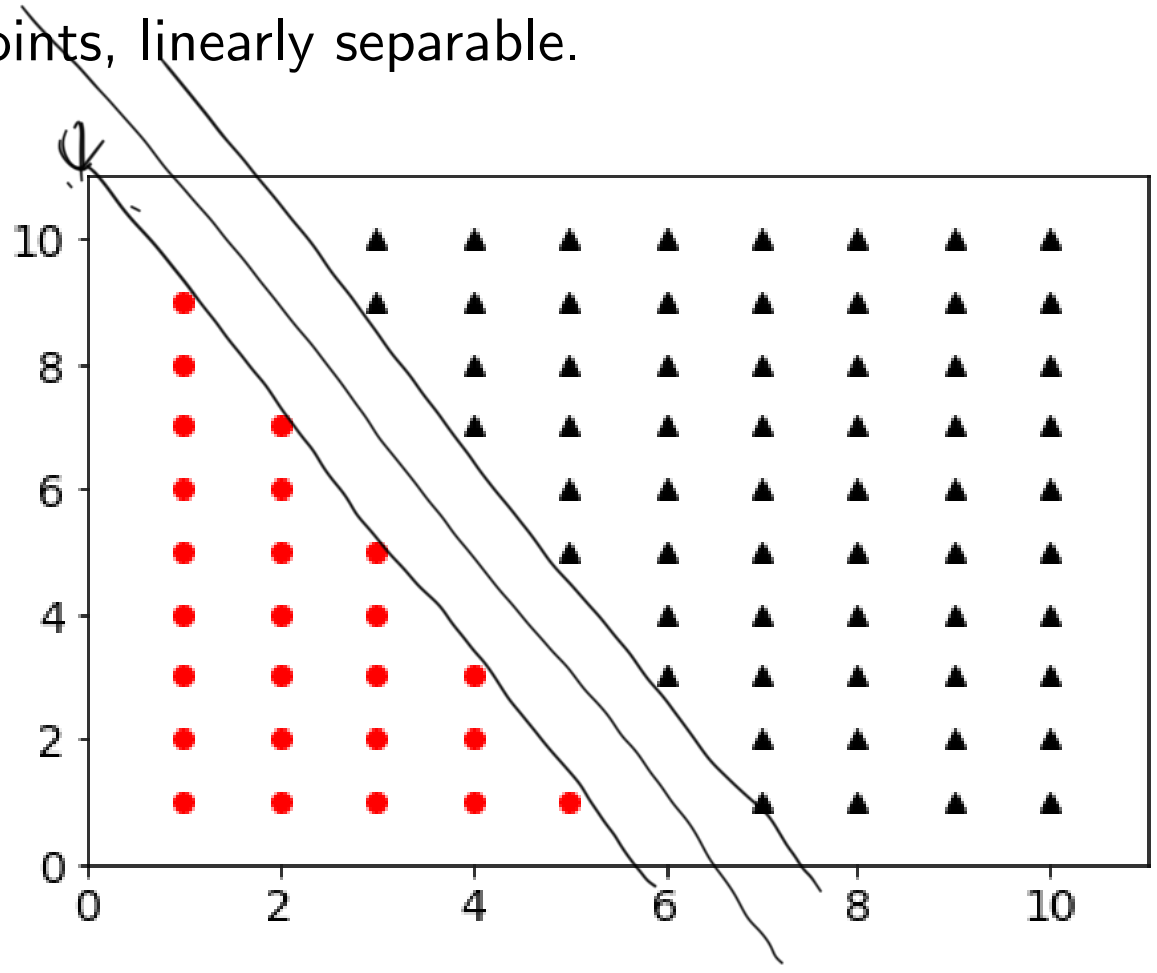


Separator: $w = -x^{(1)} + x^{(6)}$

The Perceptron in action

85 data points, linearly separable.



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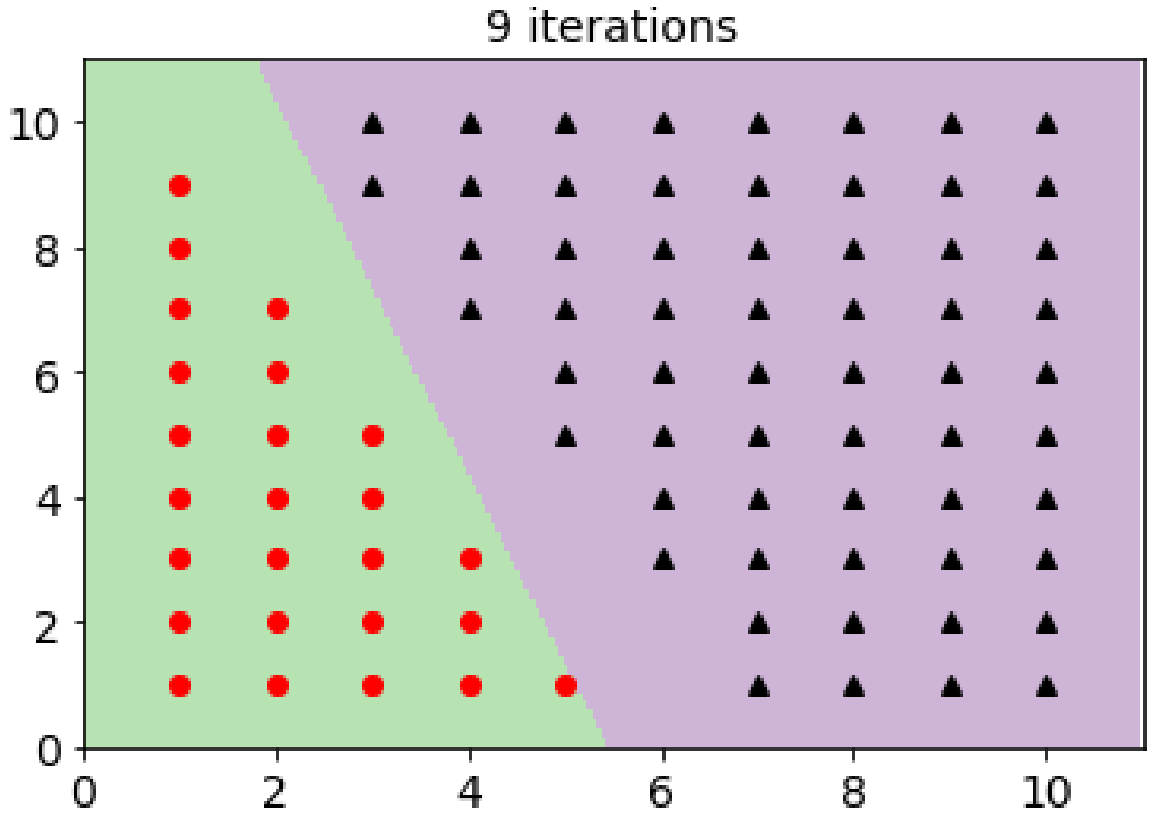

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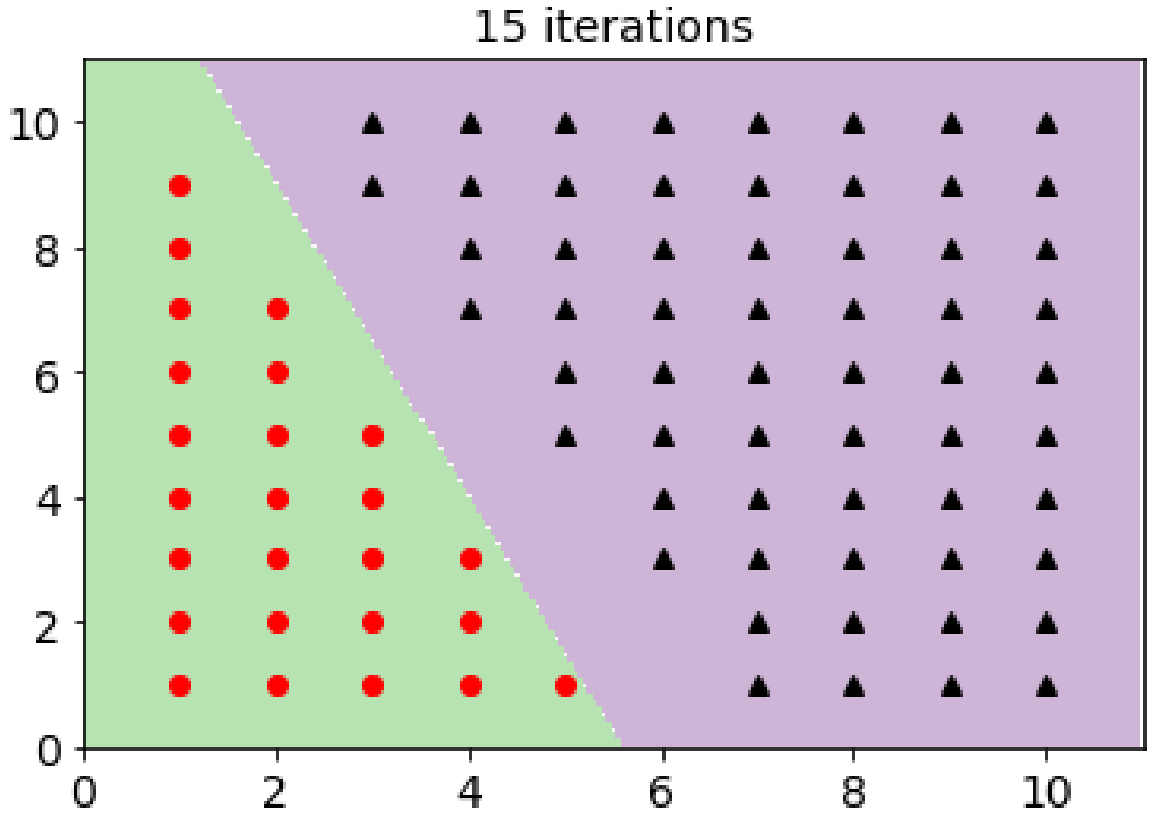

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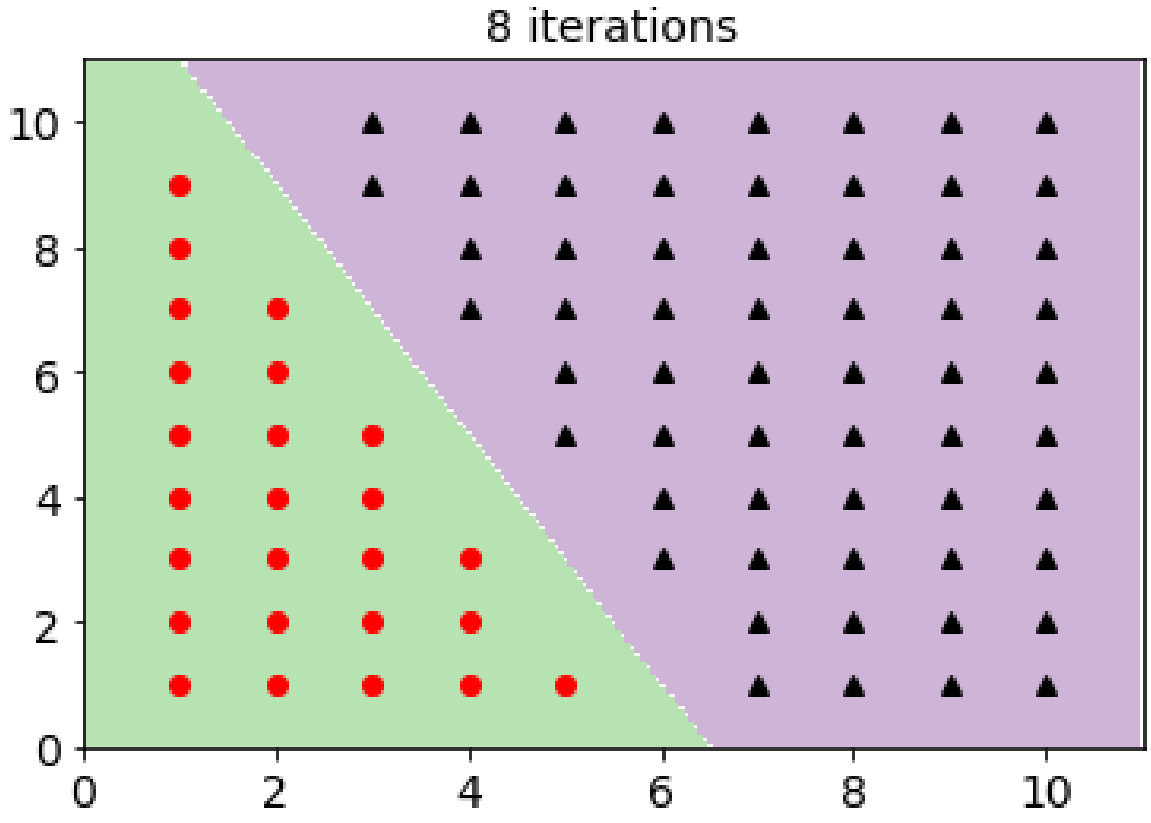

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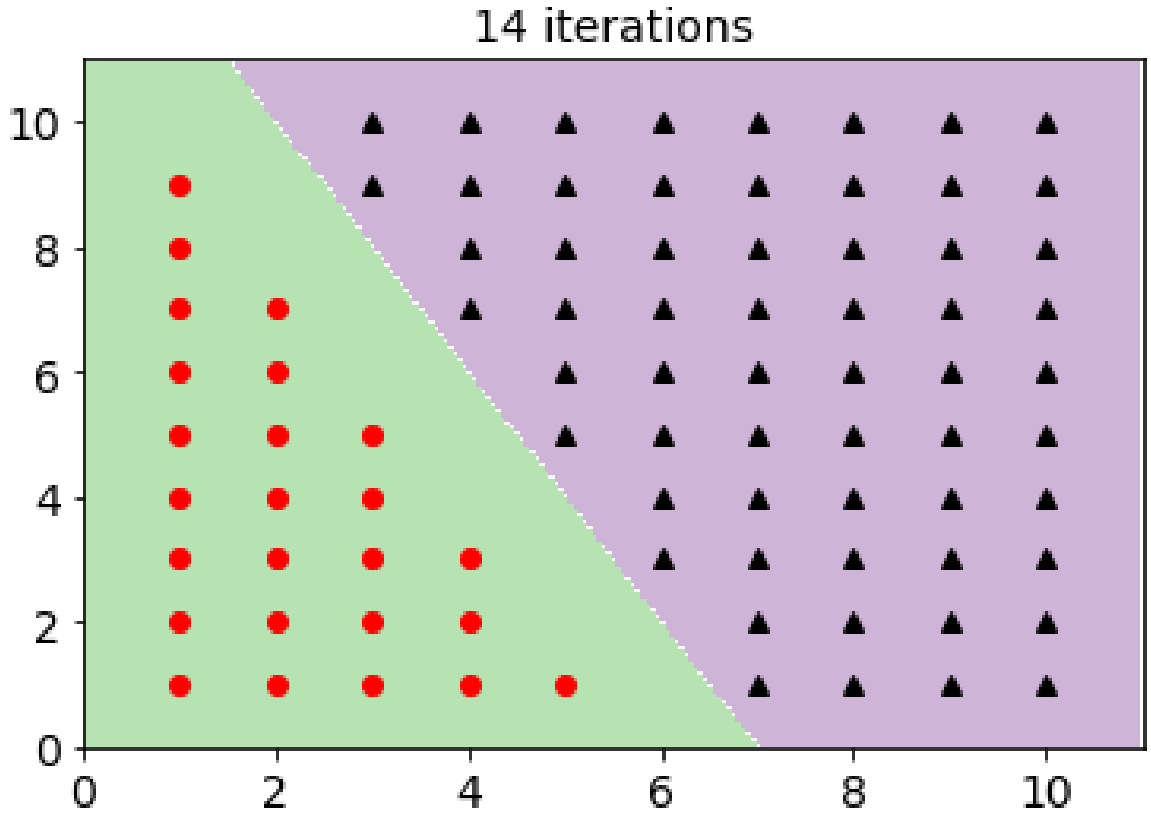

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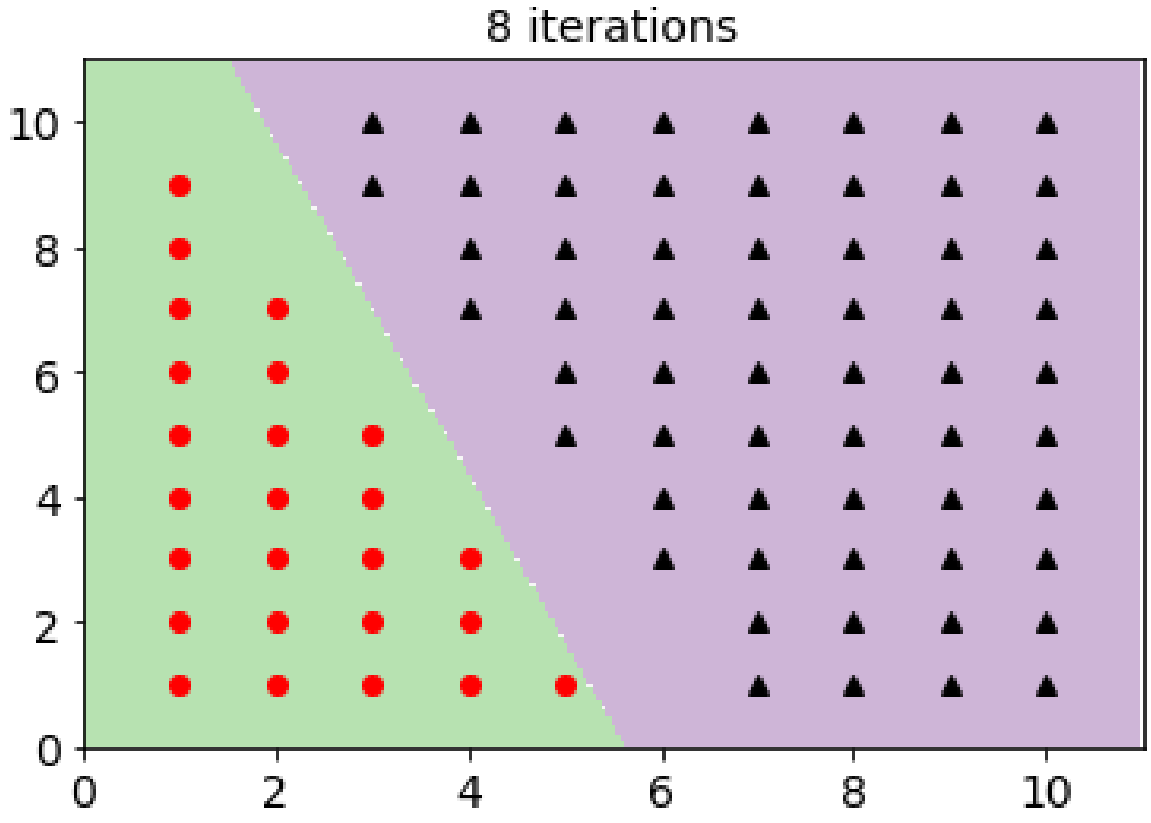
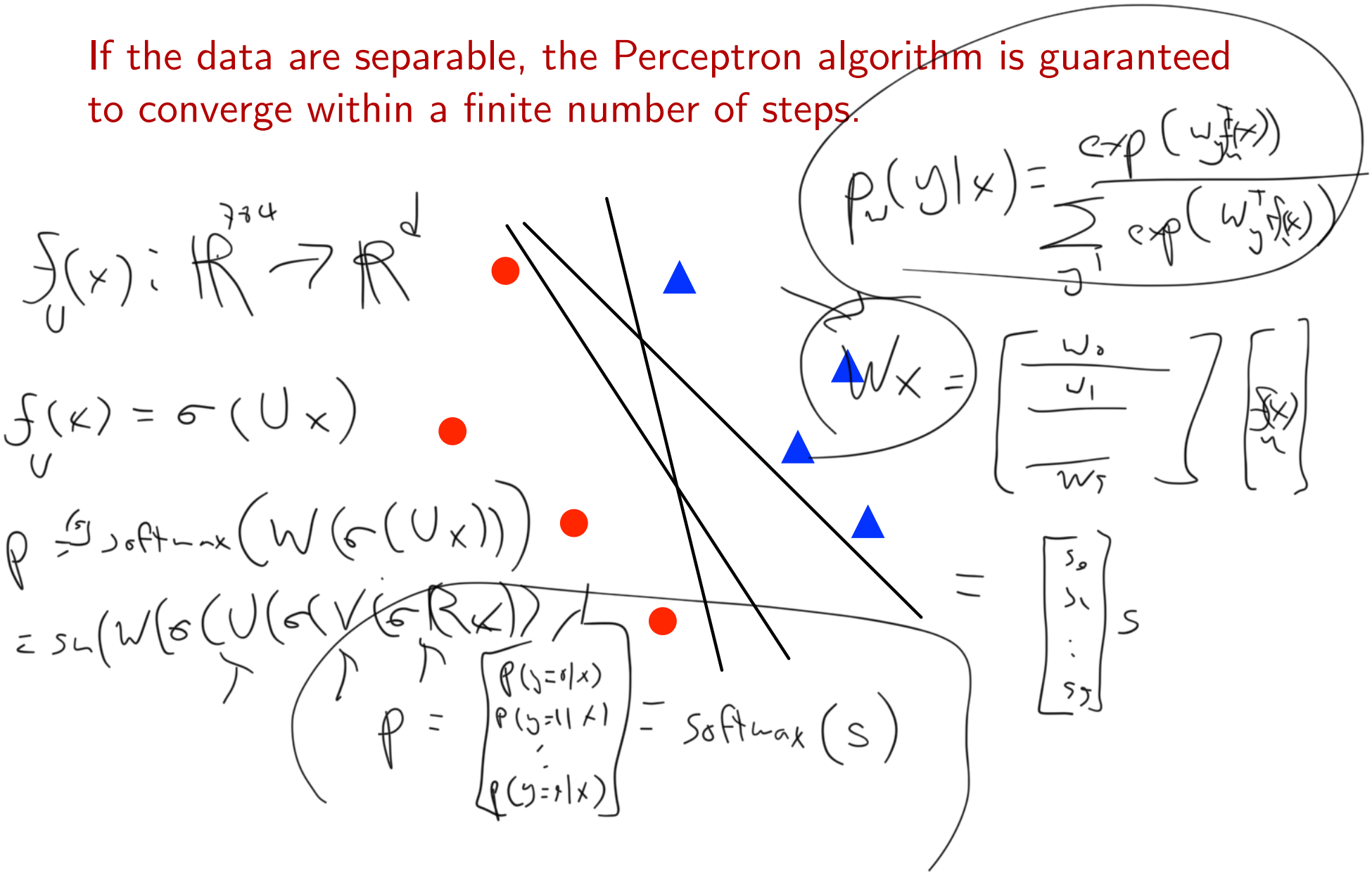


Perceptron: convergence

If the data are separable, the Perceptron algorithm is guaranteed to converge within a finite number of steps.



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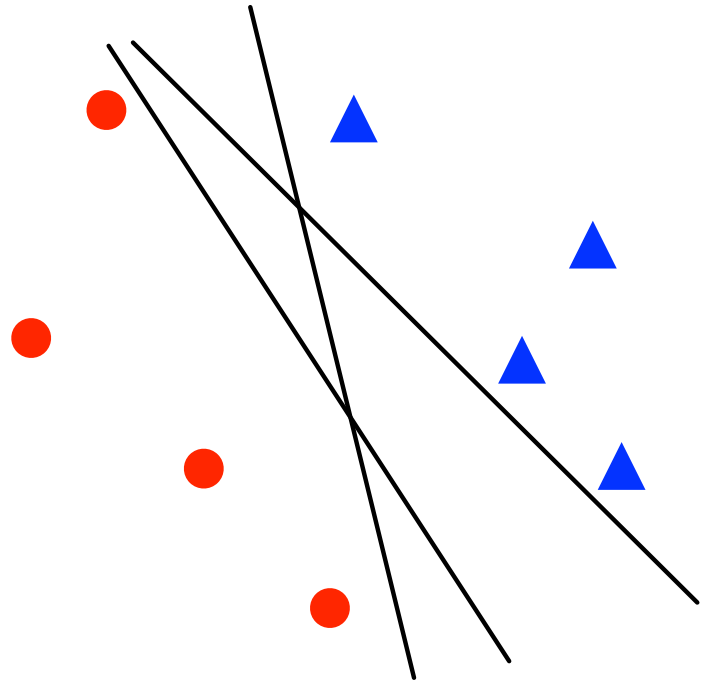

But is there a better, more systematic choice of separator? The one with the most buffer around it, for instance?