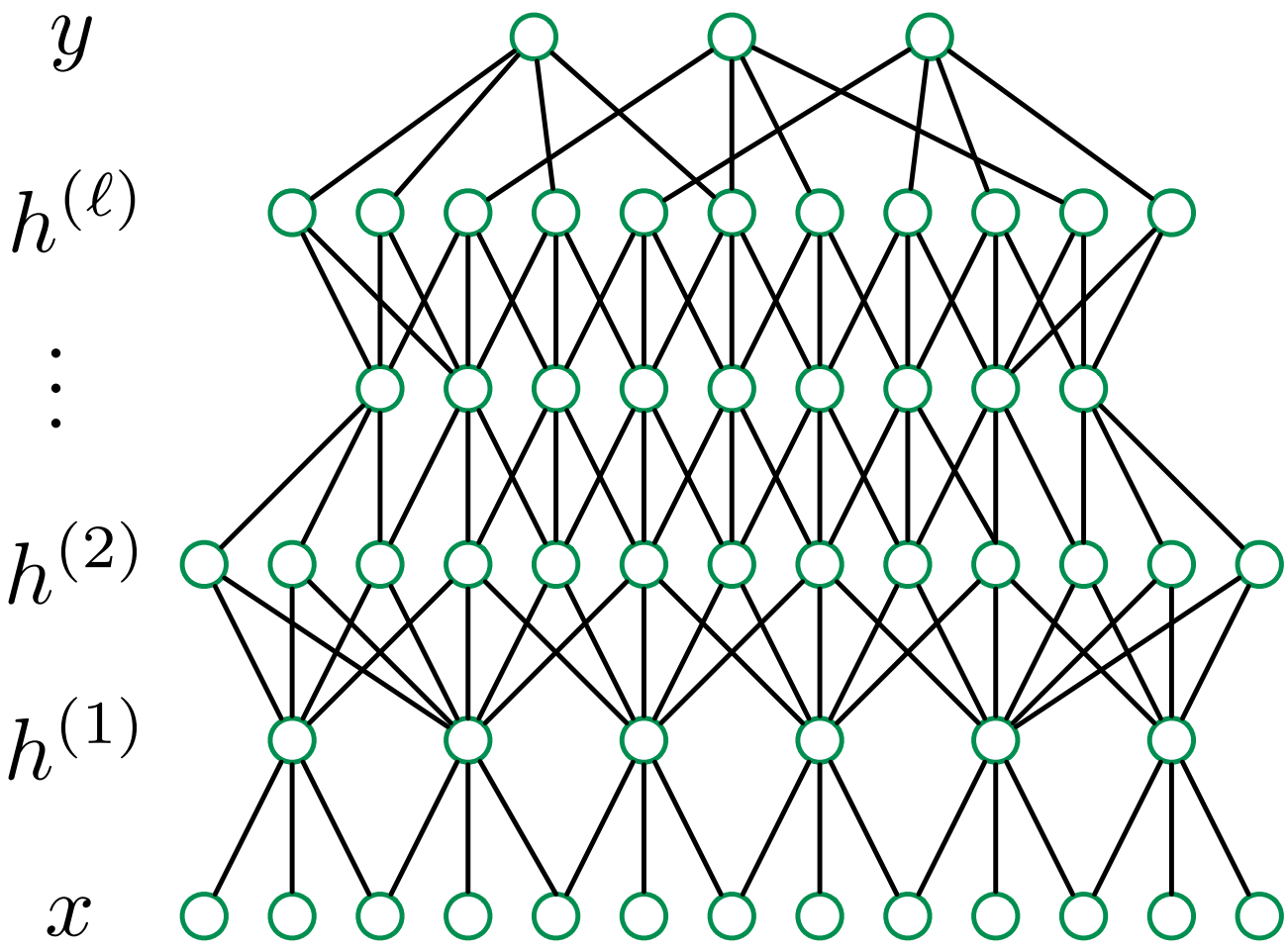
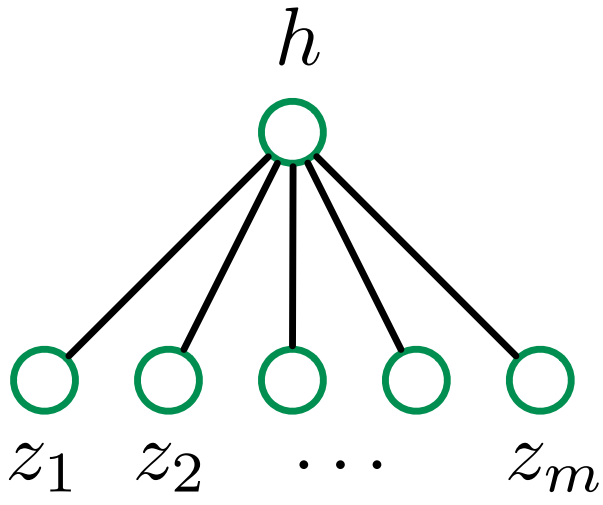
Feedforward neural nets

The architecture



The value at a hidden unit



How is h computed from $z_1, \ldots, z_m$?

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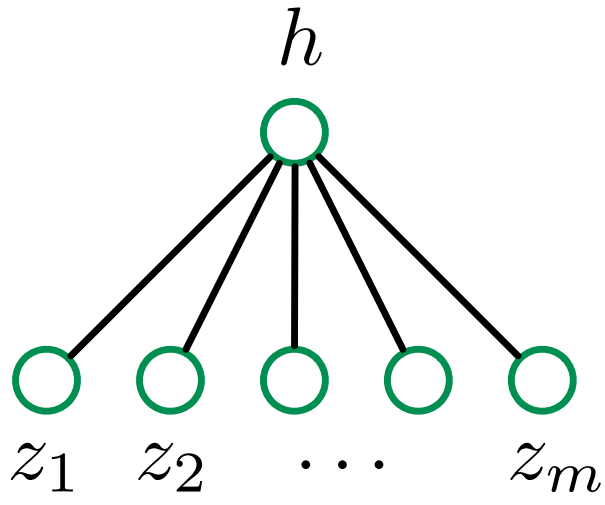
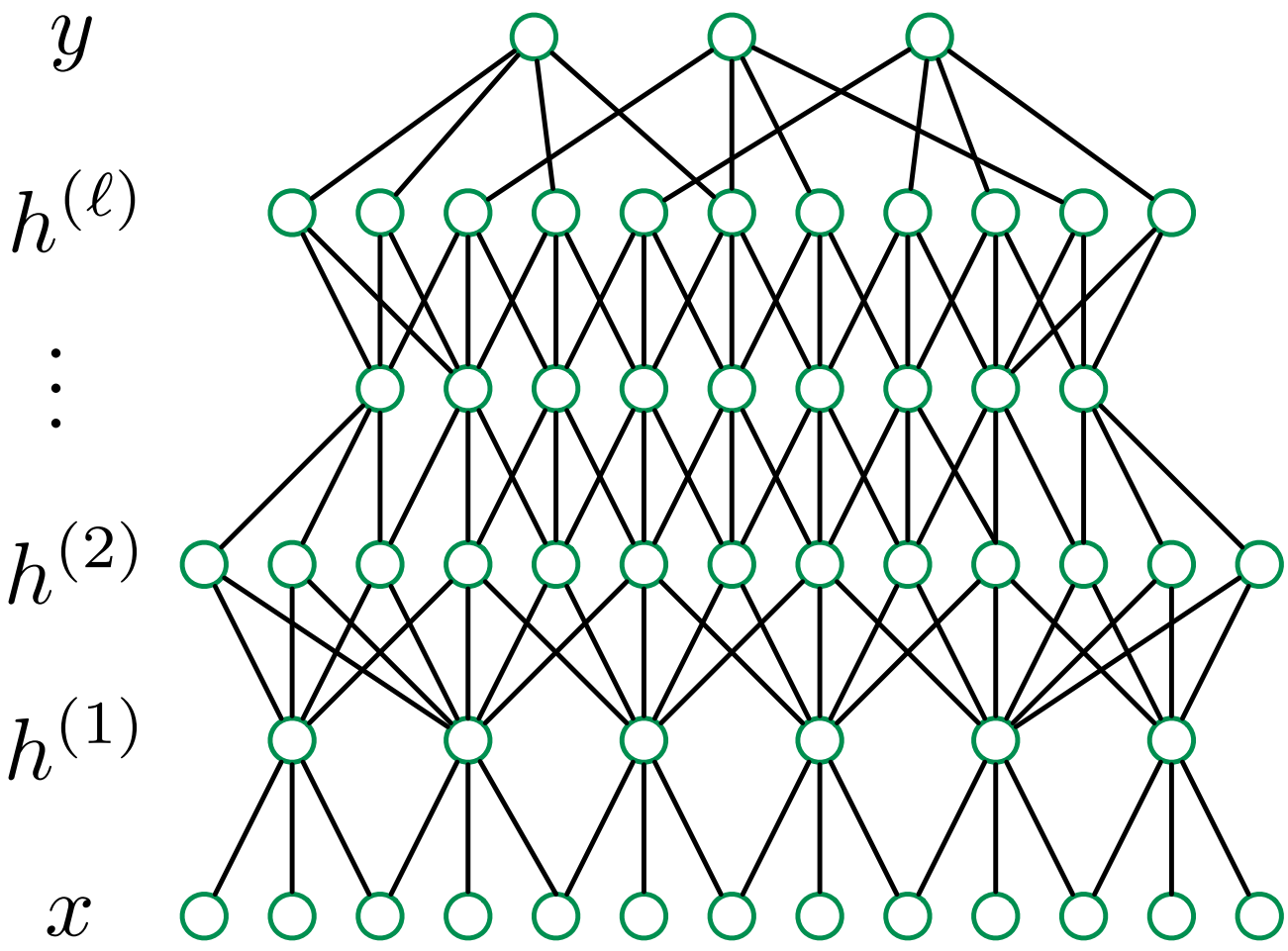


How is h computed from $z_1, \ldots, z_m$?

- $h = \sigma(w_1 z_1 + w_2 z_2 + \cdots + w_m z_m + b)$
- $\sigma(\cdot)$ is a nonlinear **activation function**, e.g. "rectified linear"

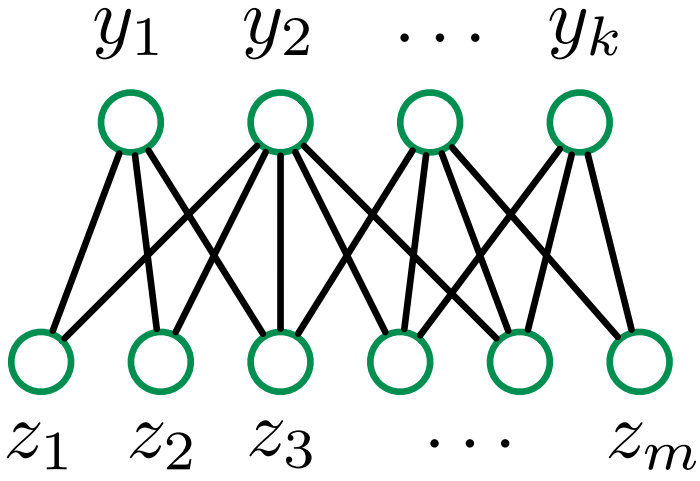
$$\sigma(u) = \begin{cases} u & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Why do we need nonlinear activation functions?



The output layer

Classification with k labels: want k probabilities summing to 1.



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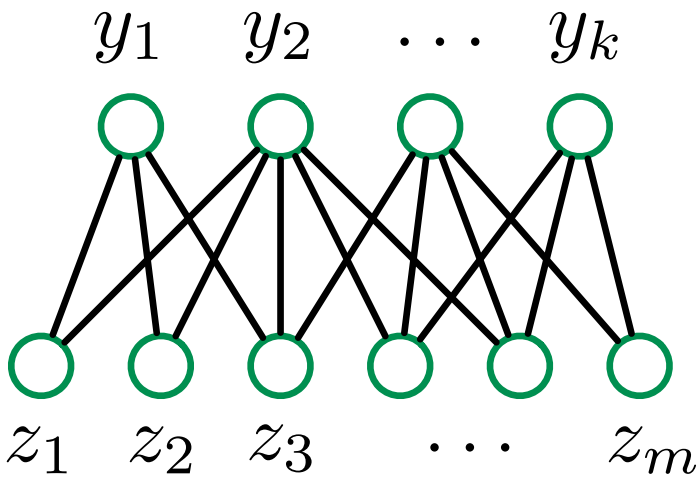
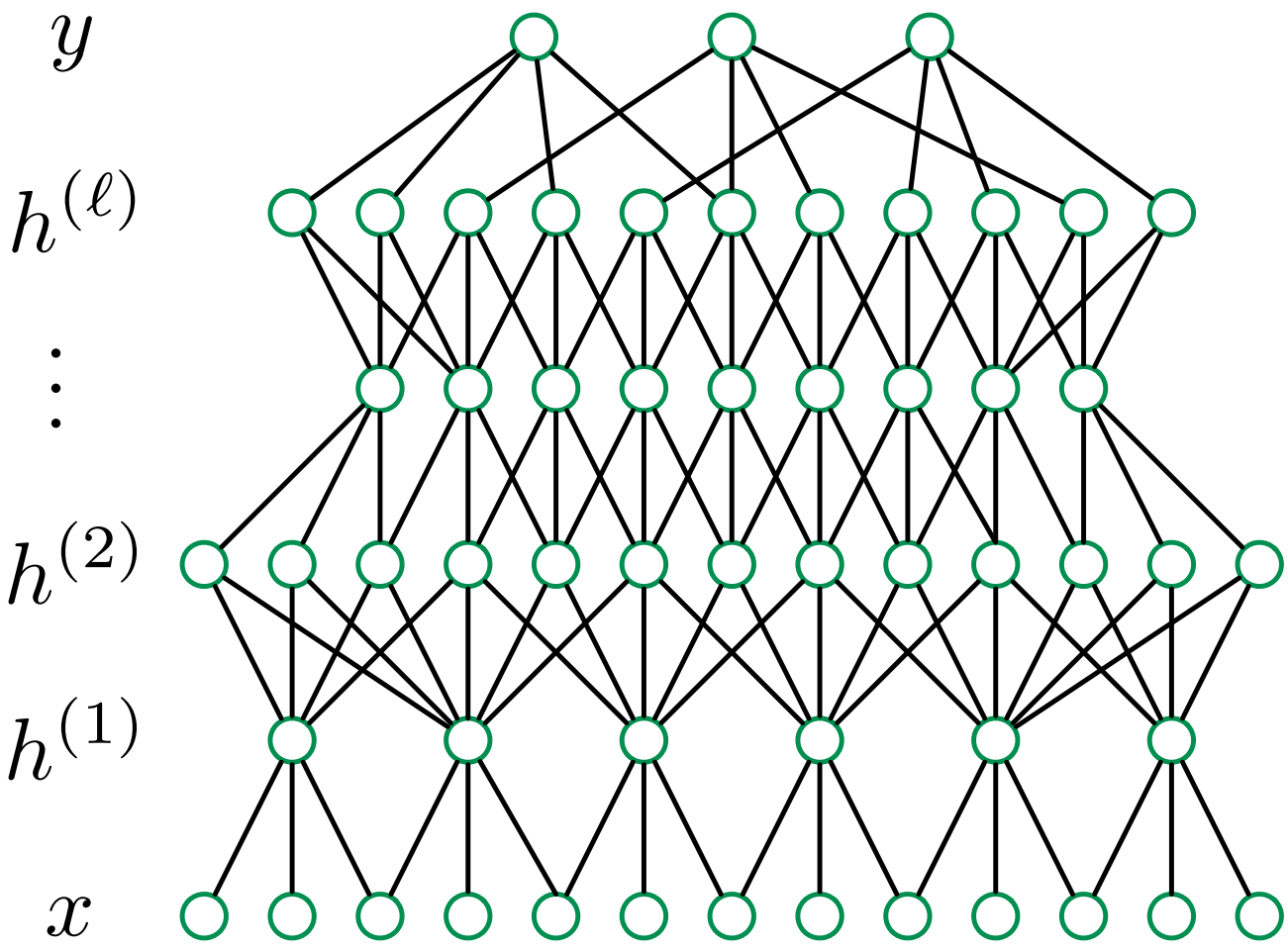


- $y_1, \ldots, y_k$ are linear functions of the parent nodes z_i.
- Get probabilities using **softmax**:

$$\Pr(\text{label } \ j) = \frac{e^{y_j}}{e^{y_1} + \cdots + e^{y_k}}.$$

The complexity



The effect of depth

- Universal approximator
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- Universal approximator
  Any function can be arbitrarily well approximated by a neural net with one hidden layer.

- Concerns about size
  To fit certain classes of functions:
  - Either: one hidden layer of enormous size
  - Or: multiple hidden layers of moderate size

Learning a net: the loss function

Classification problem with k labels.

- Parameters of entire net: W
- For any input x, net computes probabilities of labels:

$$\Pr_W(\text{label} = j|x)$$

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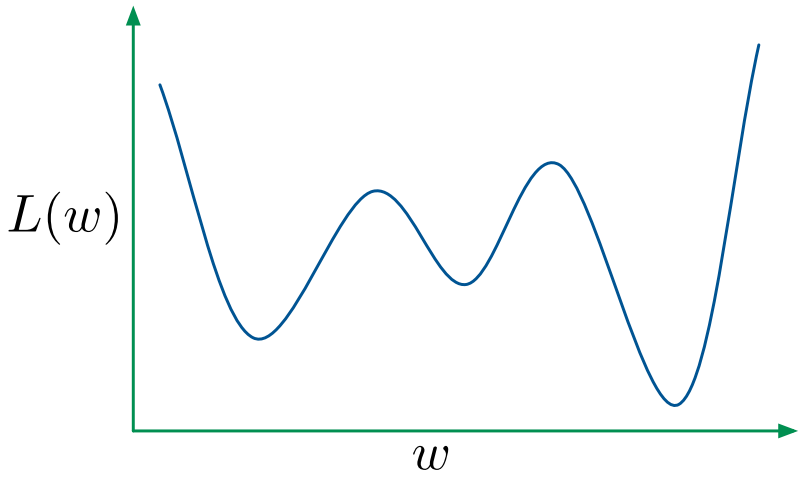
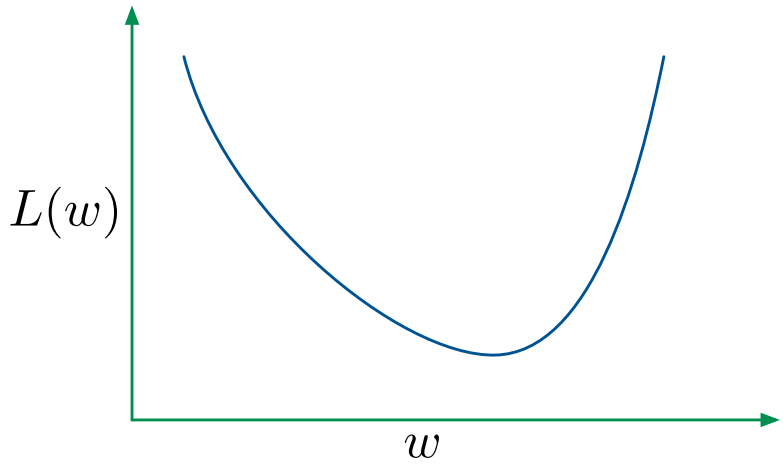
- Given data set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$, loss function:

$$L(W) = -\sum_{i=1}^{n} \ln \Pr_W(y^{(i)}|x^{(i)})$$

(sometimes called **cross-entropy**).

Nature of the loss function





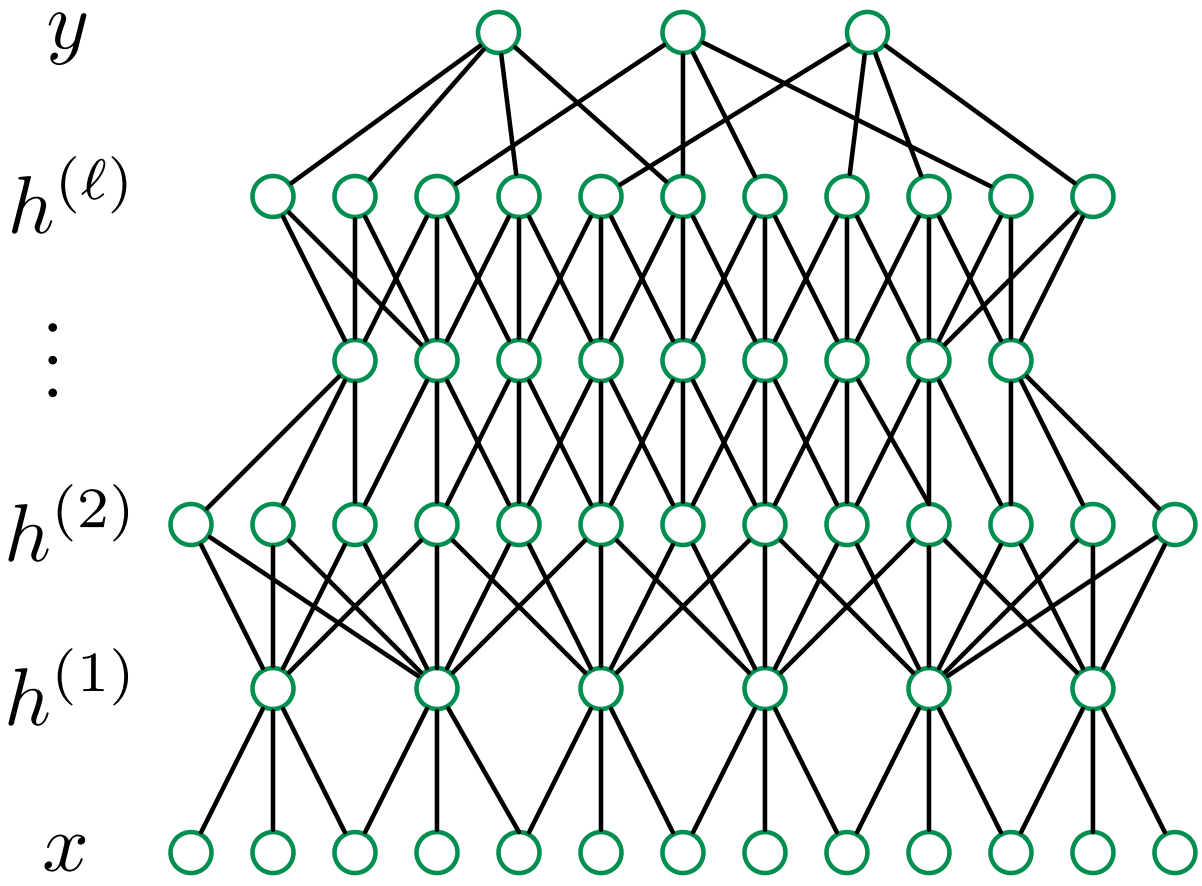
Variants of gradient descent

Initialize W and then repeatedly update.

1. Gradient descent
   Each update involves the entire training set.
2. Stochastic gradient descent
   Each update involves a single data point.
3. Mini-batch stochastic gradient descent
   Each update involves a modest, fixed number of data points.

Derivative of the loss function

Update for a specific parameter: derivative of loss function wrt that parameter.



Chain rule

1. Suppose $h(x) = g(f(x))$, where $x \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$.

   Then: $h'(x) = g'(f(x))\, f'(x)$

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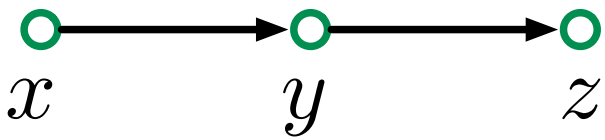
2. Suppose z is a function of y, which is a function of x.

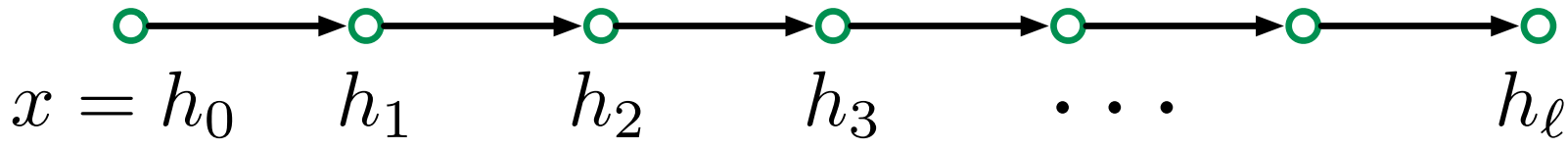


Then:

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}$$

A single chain of nodes

A neural net with one node per hidden layer:



For a specific input x,

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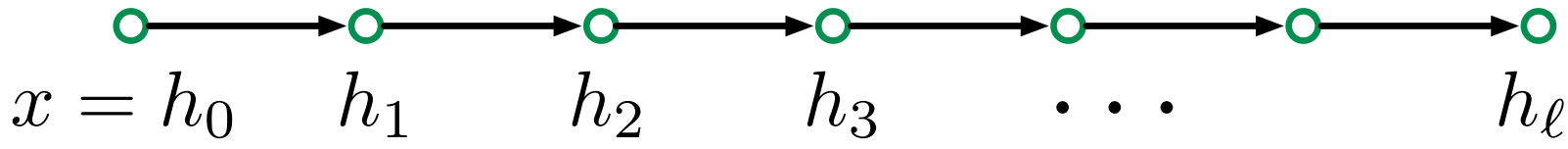

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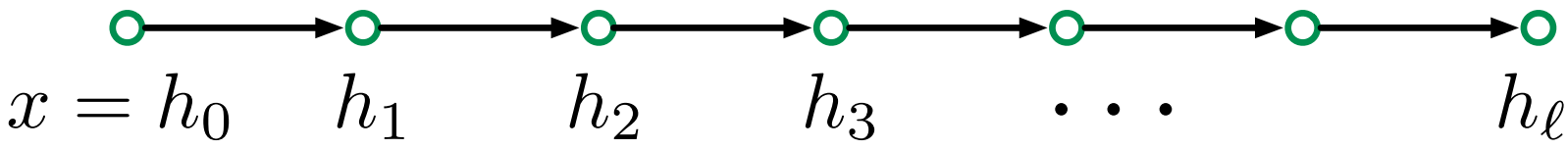
To compute dL/dw_i we just need dL/dh_i:

$$\frac{dL}{dw_i} = \frac{dL}{dh_i}\frac{dh_i}{dw_i} = \frac{dL}{dh_i}\sigma'(w_i h_{i-1} + b_i)\, h_{i-1}$$

Backpropagation

- On a single forward pass, compute all the h_i.
- On a single backward pass, compute $dL/dh_\ell, \ldots, dL/dh_1$



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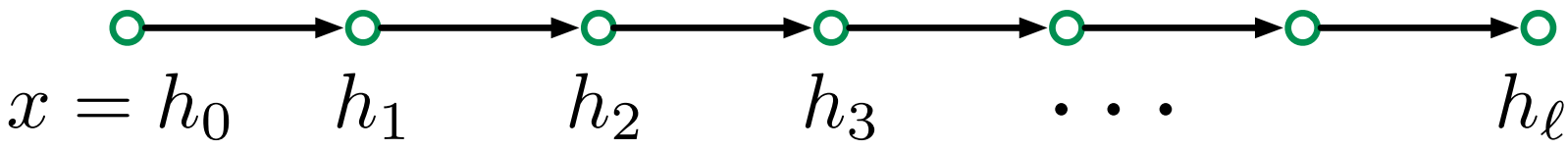

From $h_{i+1} = \sigma(w_{i+1}h_i + b_{i+1})$, we have

$$\frac{dL}{dh_i} = \frac{dL}{dh_{i+1}} \frac{dh_{i+1}}{dh_i} = \frac{dL}{dh_{i+1}} \sigma'(w_{i+1}h_i + b_{i+1})\, w_{i+1}$$

Improving generalization

1. **Early stopping**
   - Validation set to better track error rate
   - Revert to earlier model when recent training hasn't improved error

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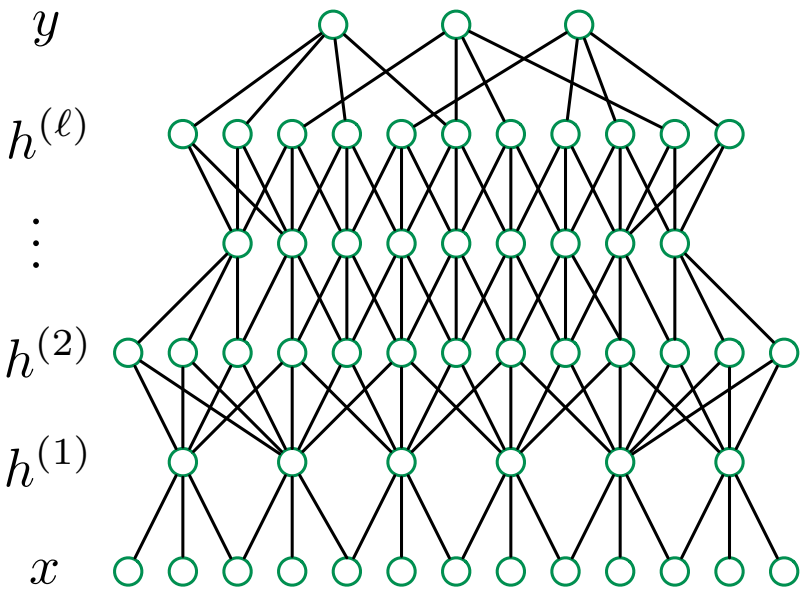
2. **Dropout**

   During training, delete each hidden unit with probability 1/2, independently.