CS 140

Class Notes 3

1 Boolean Axioms

The list of Boolean algebra **axioms** for any arbitrary boolean algebra B are as follows:

Axiom 1 (Closure property):

- (a) B is closed with respect to the operator +;
- (b) B is also closed with respect to the operator $\bullet$;

Axiom 2 (Identity element):

- (a) B has an identity element i with respect to $\bullet$, such that B $\bullet$ i = B;
- (b) B also has an identity element j with respect to +, such that B + j = B;

Axiom 3 (Commutativity Property):

- (a) B is commutative with respect to +;
- (b) B is also commutative with respect to $\bullet$;

Axiom 4 (Distributivity Property):

- (a) The operator $\bullet$ is distributive over +;
- (b) Similarly, the operator + is distributive over $\bullet$;

Axiom 5 (Complement Element): For every $x \in B$, there exists an element $x' \in B$, the complement of x, such that:

- (a) $x + x' = i$, where i is the identity element with respect to the $\bullet$ operator, and
- (b) $x \bullet x' = j$, where j is the identity element with respect to the + operator.

Axiom 6 (Cardinality Bound): There are at least two elements $x, y \in B$ such that $x \neq y$.

An interesting observation on the Boolean axioms just listed is that if you effect the following two transformations on any statement in the axiomatic base of Boolean algebra, you will generate a statement that continues to be part of the axiomatic base.

- Interchange + with $\bullet$
- Complement all variables and constants

This important observation leads to the *Duality Principle* of Boolean Algebra, which states that in any Boolean Algebra any expression retains its truth value when you effect the listed two transformations.

For the *two-valued* Boolean algebra that we will be working with, the duality principle can be concretized equivalently as:

- Interchange + with •
- Interchange 0 with 1
- Complement all uncomplemented variables, while uncomplementing complemented variables.