Linear algebra and geometric transformations in 2D

Computer Graphics

CSE 167

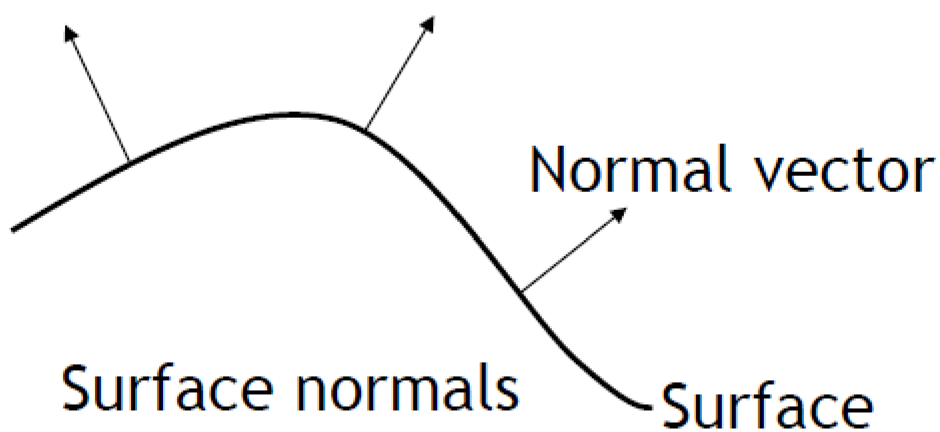
Lecture 2

CSE 167: Computer Graphics

- Linear algebra
  - Vectors
  - Matrices
- Points as vectors
- Geometric transformations in 2D
  - Homogeneous coordinates

Vectors

- Represent magnitude and direction in multiple dimensions
- Examples
  - Translation of a point
  - Surface normal vectors (vectors orthogonal to surface)



Based on slides courtesy of Jurgen Schulze

Vectors and arithmetic

Examples using 3-vectors

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Vectors are column vectors

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix} \qquad \mathbf{a} - \mathbf{b} = \begin{bmatrix} a_x - b_x \\ a_y - b_y \\ a_z - b_z \end{bmatrix}$$

Vectors must be the same length

$$-\mathbf{a} = \begin{bmatrix} -a_x \\ -a_y \\ -a_z \end{bmatrix} \qquad s\,\mathbf{a} = \begin{bmatrix} sa_x \\ sa_y \\ sa_z \end{bmatrix}$$

where s is a scalar

Magnitude of a vector

- The magnitude of a vector is its norm

Example using 3-vector

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- A vector if magnitude 1 is called a unit vector
- A vector can be unitized by dividing by its norm

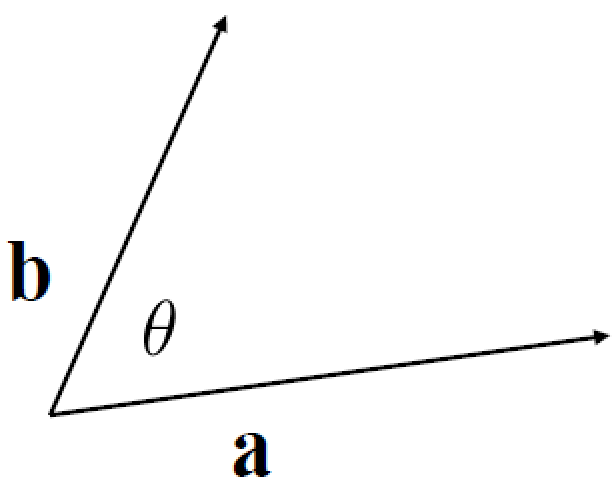
$$\frac{\mathbf{v}}{|\mathbf{v}|}$$

Dot product of two vectors

$$\mathbf{a} \cdot \mathbf{b} = \sum a_i b_i$$

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

$$\mathbf{a} \cdot \mathbf{b} = |a||b|\cos\theta$$



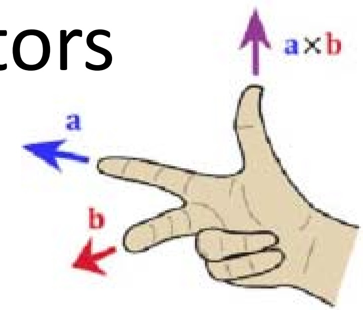
Angle between two vectors

$$\cos\theta = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right)$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right)$$

Cross product of two 3-vectors

- The cross product of two 3-vectors **a** and **b** results in another 3-vector that is orthogonal (using right hand rule) to the two vectors



$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

Cross product of two 3-vectors

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

Matrices

- 2D array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix addition

- Matrices must be the same size

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \dots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} & \dots & a_{2,n}+b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}+b_{m,1} & a_{2,2}+b_{2,2} & \dots & a_{m,n}+b_{m,n} \end{bmatrix}$$

- Matrix subtraction is similar

Matrix-scalar multiplication

$$s\mathbf{M} = \mathbf{M}s = \begin{bmatrix} sm_{1,1} & sm_{1,2} & \ldots & sm_{1,n} \\ sm_{2,1} & sm_{2,2} & \ldots & sm_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ sm_{m,1} & sm_{2,2} & \ldots & sm_{m,n} \end{bmatrix}$$

Matrix-matrix multiplication

$$\mathbf{AB} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$

$$(\mathbf{AB})_{i,j} = \mathbf{C}_{i,j} = \sum_{k=1}^{q} a_{i,k} b_{k,j}, \quad i \in 1..p, j \in 1..r$$

Matrix-vector multiplication

- Same as matrix-matrix multiplication
  - Example: 3x3 matrix multiplied with 3-vector

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$

Transpose

- $\mathbf{A}^\mathrm{T}$ is the matrix $\mathbf{A}$ flipped over its diagonal
  - Example $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^\mathrm{T} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$
- Vectors can also be transposed to convert between column and row vectors
  - Example $\begin{bmatrix} 1 & 2 \end{bmatrix}^\mathrm{T} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The identity matrix

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{MI} = \mathbf{IM} = \mathbf{M}, \quad \text{for any } \mathbf{M} \in \mathbf{R}^{n \times n}$$

Matrix inverse

- The inverse of a square matrix **M** is a matrix **M**$^{-1}$ such that

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

- A square matrix has an inverse if and only if its determinant is nonzero
- The inverse of a product of matrices is

Example using three matrices

$$(\mathbf{MPQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\mathbf{M}^{-1}$$

Representing points using vectors

- 2D point

$$\mathbf{x} = (x, y)^{\mathsf{T}}$$

- 3D point

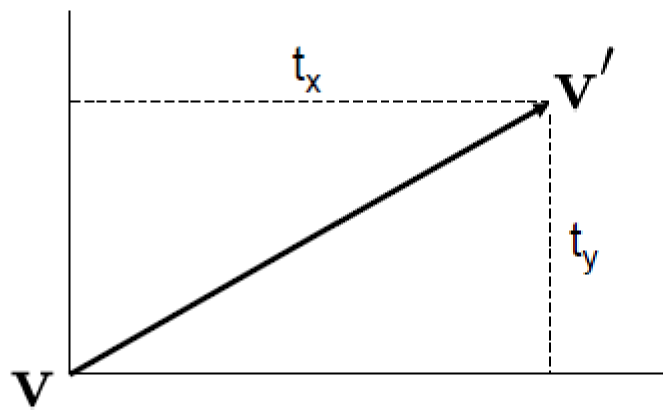
$$\mathbf{X} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z})^{\mathsf{T}}$$

Geometric transformations in 2D

- Operations on vectors (or points)
  - Translation
  - Linear transformation
    - Scale
    - Shear
    - Rotation
    - Any combination of these
  - Affine transformation
    - Linear transformation followed by translation

2D translation

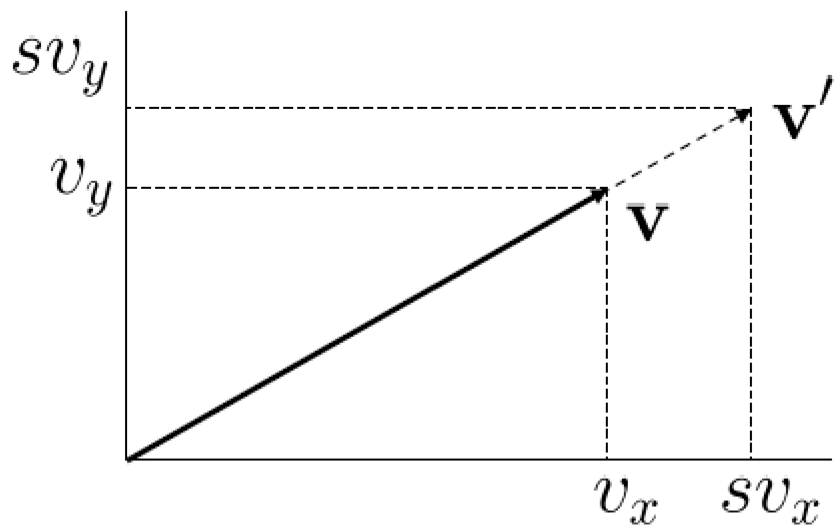
- Translation of vector **v** to **v'** under translation **t**



$$v' = \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

2D uniform scale

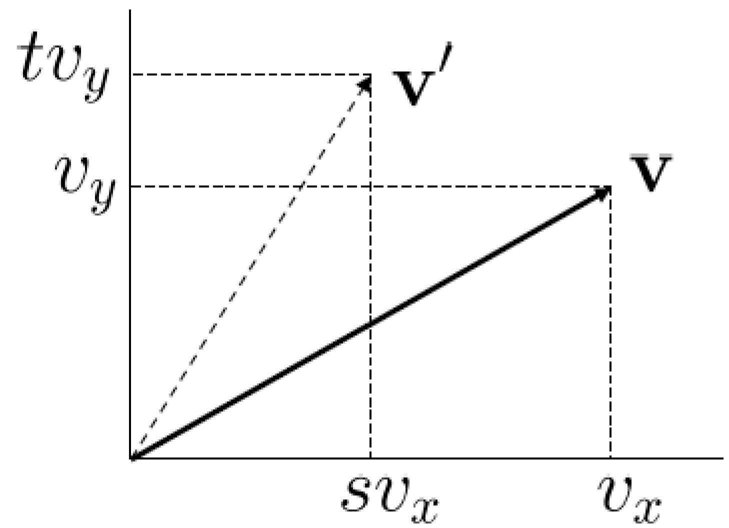
- Scale x and y the same



$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \mathbf{v}'$$

2D nonuniform scale

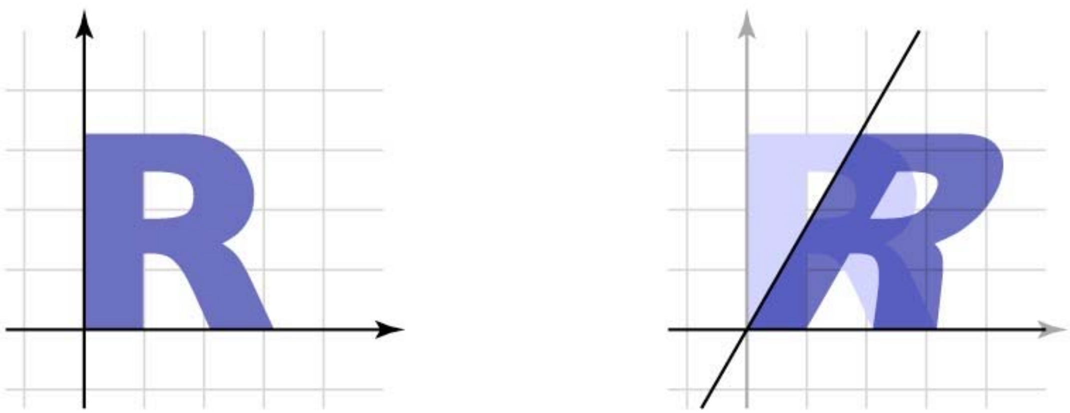
- Scale x and y independently



$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \mathbf{v}'$$

2D shear

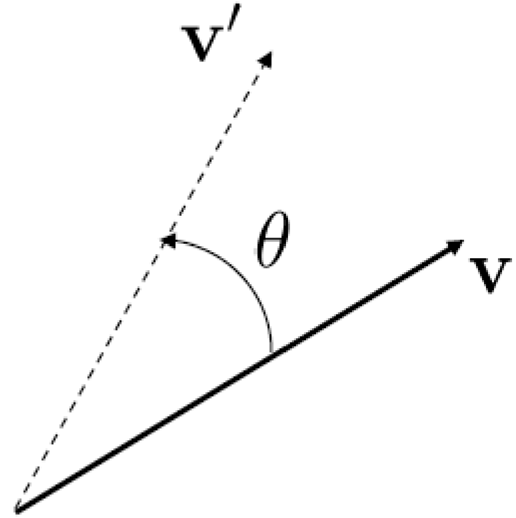
- Shear in x direction (horizontal)



$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x + ay\\y\end{bmatrix}$$

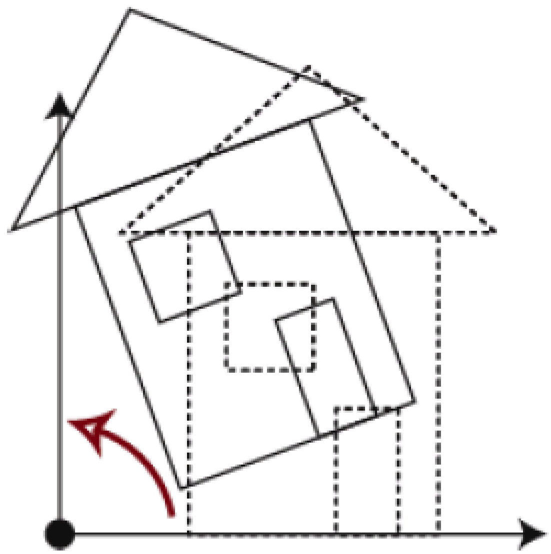
2D rotation

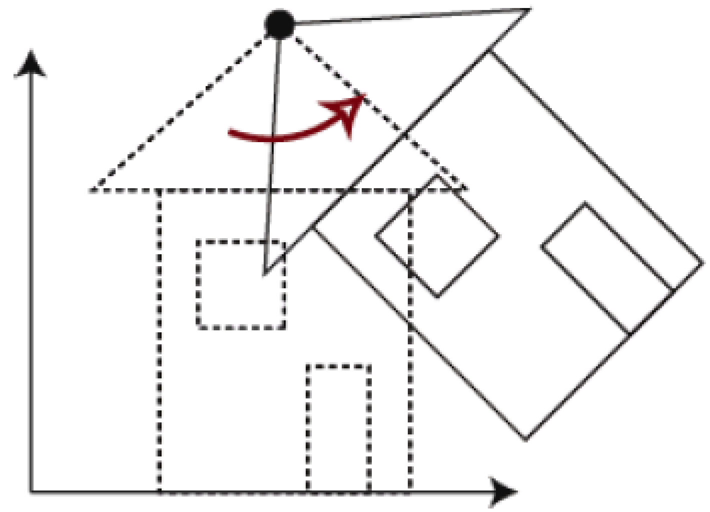
- Positive angles rotate counterclockwise



$$\mathbf{v}' = \mathbf{R}(\theta)\mathbf{v} \quad \text{where} \quad \mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2D rotation about a point

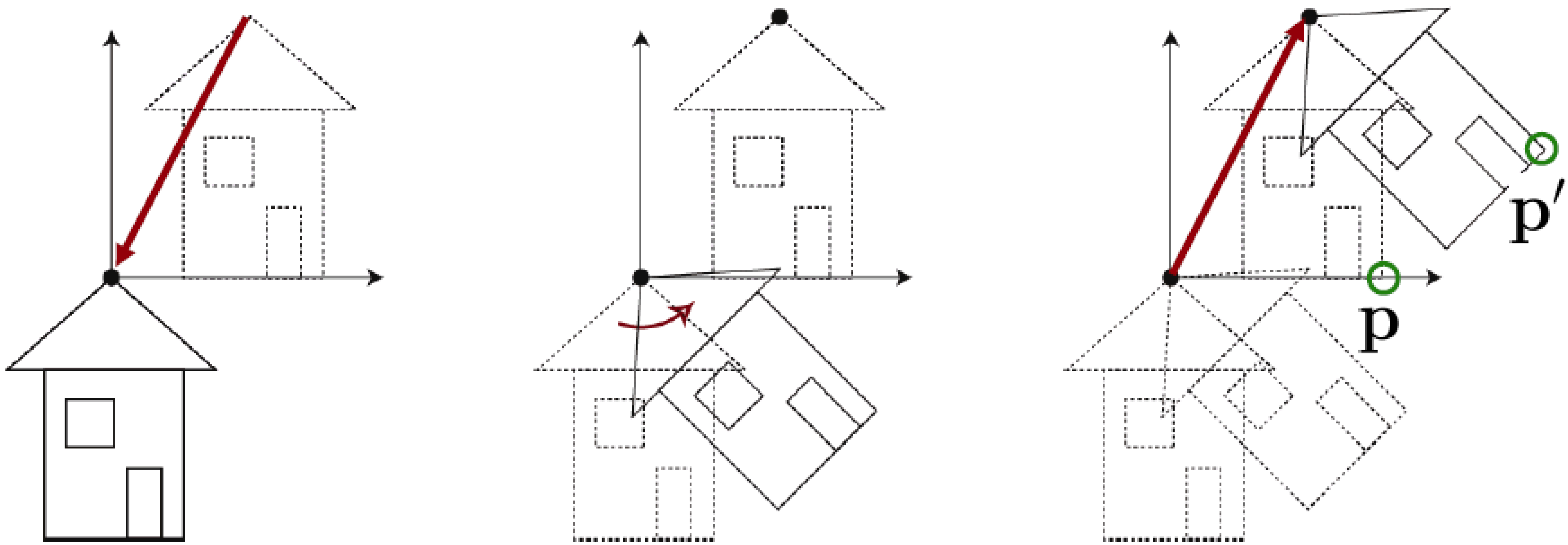




Rotation around origin:
p’ = R p

Rotation around pivot point:
p’ = ?

2D rotation about a point



1. Translate point to the origin

2. Rotate about the origin

3. Translate origin back to point

2D rotation about a point

- This can be accomplished with one transformation matrix, if we use homogeneous coordinates
- A 2D point using affine homogeneous coordinates is a 3-vector with 1 as the last element

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D translation using homogeneous coordinates

- 2D translation using a 3x3 matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Inverse of 2D translation is inverse of 3x3 matrix

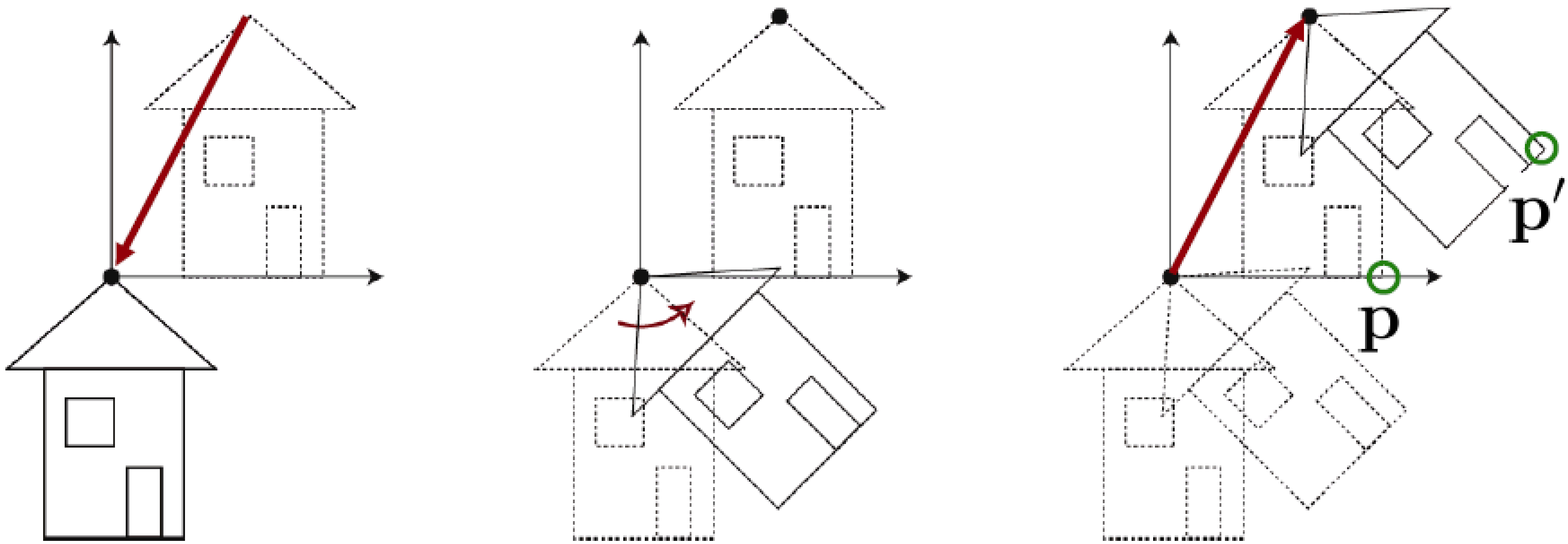
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

2D rotation using homogeneous coordinates

- 2D rotation using homogenous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D rotation about a point using homogeneous coordinates



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Important: transformation matrices are applied right to left

2D rotation about a point using homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where $\mathbf{M} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$