CSE 105
THEORY OF COMPUTATION

"Winter" 2018

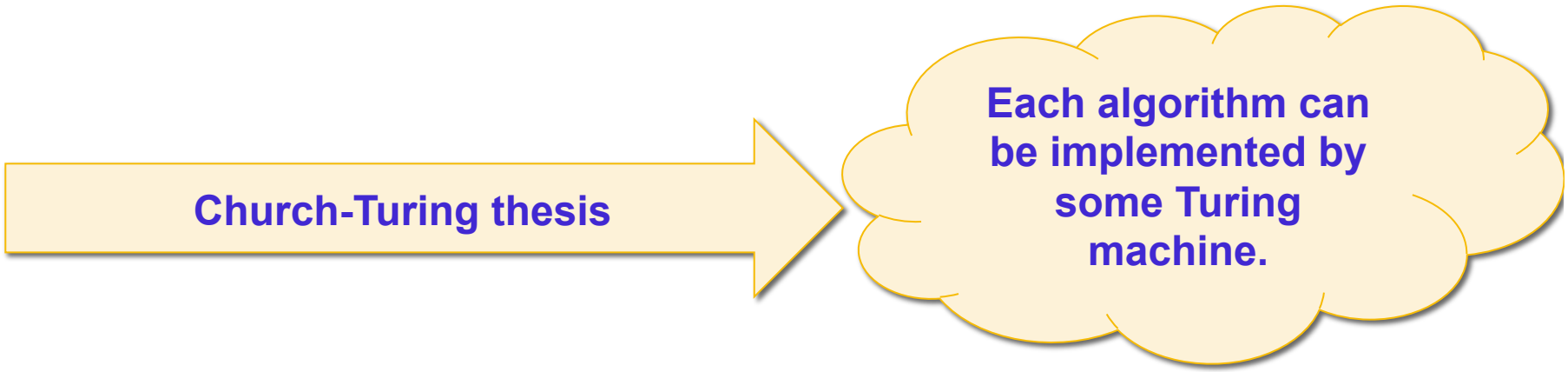
http://cseweb.ucsd.edu/classes/wi18/cse105-ab/

Today's learning goals

Sipser Section 4.1

- Explain what it means for a problem to be decidable.
- Justify the use of encoding.
- Give examples of decidable problems.

High-level description = Algorithm



Algorithms

So far: machines describing / recognizing sets.

What questions can we answer?

- Is a string a palindrome?
- Does a string have even length?

Answering these questions is the same as describing the set of strings for which the answer is yes.

Computational problems

Sipser p. 194

A computational problem is **decidable** iff the language encoding the problem instances is decidable

- Does a specific DFA accept a given string? encoded by
  { representations of DFAs M and strings w such that w is in L(M) }
- Is the language generated by a specific CFG empty? encoded by
  { representations of CFGs G such that L(G) = Ø }
- Is a Turing machine a decider? encoded by
  { representations of Turing machines M such that M always halts }

Representations for computational problems

Sipser p. 185

To decide these problems, we need to represent the objects of interest as **strings**

For inputs that aren't strings, we have to **encode the object** (represent it as a string) first

To define TM M:

"On input w …"

1. ..
2. ..
3. …

Notation:

<O> is the **string** that represents (encodes) the object O

$\langle O_1, \ldots, O_n \rangle$ is the single string that represents the tuple of objects $O_1, \ldots, O_n$

$DFA = (\{q_0, q_1\}, \{0,1\}, (q_0, 1, q_1), (q_1, 1, q_0), q_1, \{q_0\})$

$\delta(q_0,0)$ $\delta(q_0,1)$ $\delta(q_1,0)$ $\delta(q_1,1)$

Representations for computational problems

Sipser p. 185

To decide these problems, we need to represent the objects of interest as **strings**

For inputs that aren't strings, we have to **encode the object** (represent it as a string) first

To define TM M:

"On input w …

1. ..
2. ..
3. …

Assumption:
There are Turing machine subroutines that can decode the string representations of common objects so we can interact with them as intended
e.g. from string representation of Turing machine, can decode $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$

Encoding inputs

Payoff: problems we care about can be reframed as languages of strings

e.g. "Recognize whether a string is a palindrome."

$\{ w \mid w \text{ in } \{0,1\}^* \text{ and } w = w^R \}$

e.g. "Check whether a string is accepted by a DFA."

$\{ \langle B,w \rangle \mid B \text{ is a DFA over } \Sigma, w \text{ in } \Sigma^*, \text{ and } w \text{ is in } L(B) \}$

e.g. "Check whether the language of a PDA is infinite."

$\{ \langle A \rangle \mid A \text{ is a PDA and } L(A) \text{ is infinite} \}$

Computational problems

Does a specific DFA accept a given string? encoded by
{ representations of DFAs M and strings w such that w is in L(M) }

Define using high-level description a Turing machine M = "On input <B,w>, where B is a DFA and w is a string:

1. Type check encoding to check input is valid type.
2. Simulate B on input w (by keeping track of states in B, transition function of B, etc.)
3. If the simulations ends in an accept state of B, *accept*. If it ends in a non-accept state of B, *reject*. "

Computational problems

- **Recall**: in high-level descriptions, can simulate (run) other Turing machines / algorithms as a subroutine of program being defined.

- **To prove decidable**: need to confirm that strings in the language are *accepted* **and** that strings not in the language are *rejected* (no looping allowed).

Computational problems

Define using high-level description a Turing machine M = "On input <B,w>, where B is a DFA and w is a string:

1. Type check encoding to check input is valid type.
2. Simulate B on input w (by keeping track of states in B, transition function of B, etc.)
3. If the simulations ends in an accept state of B, *accept*. If it ends in a non-accept state of B, *reject*. "

Why is M a decider? $L(M) = A_{DFA}$

Computational problems

Vocabulary

A_{XX} "Is a given string accepted by a given machine of type XX?"

{ <B,w> | B is a XX over Σ, w in Σ*, and w is in L(B) }

E_{XX} "Is the language of a machine of type XX is empty?"

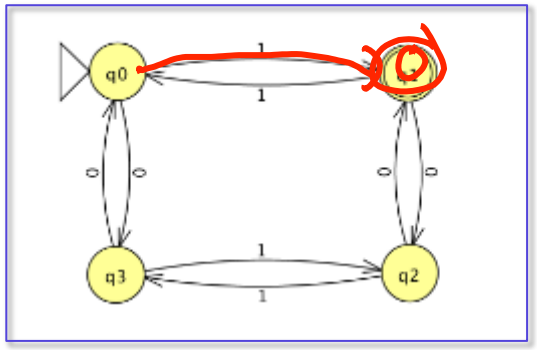
{ <A> | A is a XX over Σ, L(A) is empty }

EQ_{XX} "Are the languages of two given machines of type XX equal?"

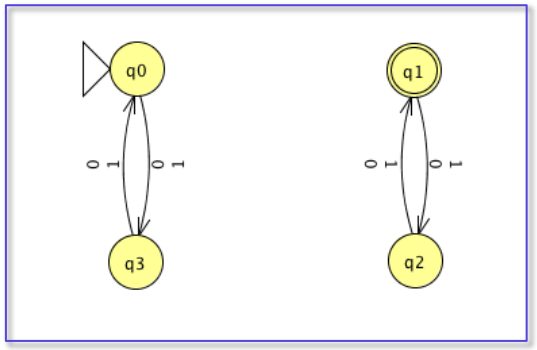
{ <A, B> | A and B are XX over Σ, L(A) = L(B)}

For DFA

M1



M2



A. <M1, 1> is in A_{DFA}
B. <M2, 01> is in A_{DFA}
C. <M1> is in E_{DFA}
D. <M1, M2> is in EQ_{DFA}
E. More than one of the above

$\langle M2 \rangle \in E_{DFA}$.

Computational problems

When the model in question is DFA: which of the following computational problems are **decidable**?

A. A_{DFA} decidable ("by construction a few slides back.")
B. E_{DFA}
C. EQ_{DFA}
D. All of the above
E. None of the above

Proving decidability

Claim: E_{DFA} is decidable

Proof: WTS that { <A> | A is a DFA over Σ, L(A) is empty } is decidable.

use DFS or BFS.

*Informally: what do you look for in the state diagram of a DFA to determine if it accepts *at least one* string?*

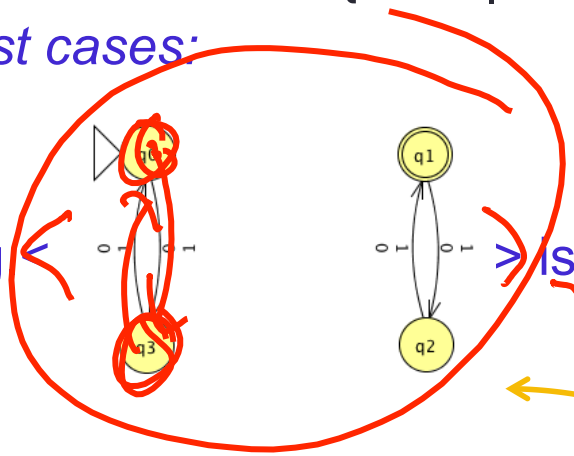
if there is a path from initial state to state in F. Then reject otherwise accept.

Proving decidability

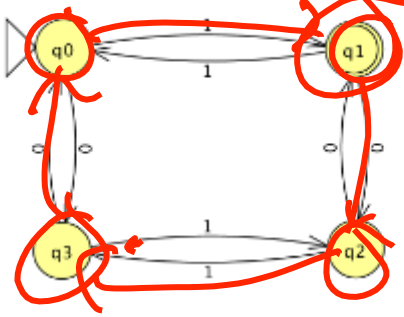
Claim: E_{DFA} is decidable

Proof: WTS that { <A> | A is a DFA over Σ, L(A) is empty } is decidable.

Test cases:



e.g. < > is in E_{DFA} ; <



> is not in E_{DFA}

accept

TM deciding E_{DFA} should accept and should reject

Proving decidability

Claim: E_{DFA} is decidable

Proof: WTS that { <A> | A is a DFA over Σ, L(A) is empty } is decidable.

Step 1: construction

Idea: breadth-first search in state diagram to look for paths to F

Proving decidability

Claim: E_{DFA} is decidable
Proof: WTS that { <A> | A is a DFA over Σ, L(A) is empty } is decidable.

Step 1: construction

Idea: breadth-first search in state diagram to look for paths to F

Define TM M_2 by: M_2 = "On input <A>:

1. Check whether input is a valid encoding of a DFA; if not, reject.
2. Mark the start state of A.
3. Repeat until no new states get marked:
   i. Loop over states of A and mark any unmarked state that has an **incoming** edge from a marked state.
4. If no final state of A is marked, *accept*; otherwise, *reject.*"

Proving decidability

Step 2: correctness proof

WTS (1) $L(M_2) = E_{DFA}$ and (2) M_2 is a decider.

(1) @ $L(M_2) \subseteq E_{DFA}$: let $\langle B \rangle \in L(M_2)$. $\Rightarrow$
there is no path from q_0 to a state in F. $\Rightarrow$
B does not accept any string $\Rightarrow$ $\langle B \rangle \in E_{DFA}$.
(b) exercise $E_{DFA} \subseteq L(M_2)$

(2) For finite graphs, BFS always halts. And so M_2 will always halt.

Proving decidability

Claim: EQ_{DFA} is decidable

Proof: WTS that { <A,B> | A and B are DFAs over Σ, L(A) = L(B)} is decidable.

Exercise

Can we use other theorem to show this?

Use the decidability of E_{DFA} and closure arguments to show EQ_{DFA} is decidable!

For next time

GroupHW5 due Saturday, February 24

For Monday, pre-class reading: Section 4.1, Theorem 4.5 (page 197) and Theorem 4.8 (page 199)